Math 5615H **Fall 2020** **Homework 7**
Posted: 10/31; due Friday, 11/6

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).**
(Baby) Rudin: Sections 3.34-3.35, 3.41-43, 3.52-3.55, 4.1-4.2.

Problem 1. Suppose that $\{a_n\}$ is a bounded sequence of real numbers and $a^* \in \mathbb{R}$. Prove that $a^* = \limsup_{n\to\infty} a_n$ **iff** for each $\varepsilon > 0$ there are at most finitely many terms of $\{a_n\}$ such that $a_n > a^* + \varepsilon$ but infinitely many terms such that $a_n > a^* - \varepsilon$. (Pay attention to the fact that it is "iff" rather than "if.")

Problem 2. Is there a real constant a such that the following series converges?

$$1 + \frac{1}{\sqrt{3}} - \frac{a}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{a}{\sqrt{4}} + \dots$$

Hint: Group the terms in triples

$$\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{a}{\sqrt{2n}}$$

and do the algebra to compare the size of each triple with $c/\sqrt{n}$ for n large enough and some c. This will work for most values of a. For other a's, do the same grouping and algebra, but the result will be different.

Problem 3. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Hint: Use the idea of a "telescoping" series.

Problem 4. Show that neither the ratio test, nor the root test applies to the series $\sum_{k=1}^{\infty} 1/\sqrt{k}$.

Problem 5. Prove the following. If the series $\sum_{k=0}^{\infty} a_k$ is convergent and $\{b_k\}_{k=0}^{\infty}$ is a monotonic convergent sequence, then the series $\sum_{k=0}^{\infty} a_k b_k$ is convergent.

Problem 6. Show that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, there are rearrangments of the series which diverge to ∞ and $-\infty$.

Problem 7. Let $f(x,y) = x^2y/(x^4+y^2)$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. *Hint*: Consider different paths of approach to $(0,0)$.