Math 5615H **Fall 2020** **Homework 9**

Posted: 11/14; Clarified Problem 4: 11/19; due Friday, 11/20

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`).
(Baby) Rudin: Sections 4.20-4.21, 4.25-4.34, 5.1-5.2.

Problem 1. If $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ a function continuous at $a \in D$ and such that $f(a) \neq 0$, then there is an interval $(a-\delta, a+\delta)$ for some $\delta > 0$ such that for all $x \in D \cap (a-\delta, a+\delta)$ the values $f(x)$ are nonzero and have the same sign as $f(a)$.

Problem 2. A function $f : X \to Y$ is called *proper*, if for any compact subset $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact. Show that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is proper if and only if $\lim |f(x)| = \infty$ as $x \to \infty$ and $x \to -\infty$.

Problem 3. Let a function $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic, that is to say, there is $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show f us uniformly continuous on $\mathbb{R}$.

Problem 4 (Guangqi Li)**.** Can a function $f : (a,b) \to \mathbb{R}$ be continuous on a bounded open interval $(a,b) \subset \mathbb{R}$ and have $\lim_{x \to b-} f(x) = \infty$? What if we require uniform continuity on (a,b)?

Problem 5. Let $f : (-1,1) \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on at 0. (It is more obviously differentiable at all other points as the product of compositions of quotients of differentiable functions.)

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Show that f is differentiable only at $x = 0$ and $f'(0) = 0$.

Problem 7. For the function in Problem 1 on the previous homework (HW 8), classify the discontinuities. That is, determine if the discontinuities at rational points are of the first or second kind. Explain your answer by stating whether the left-hand and right-hand limits at rational points exist and, if they do, what they are, as in Examples 4.27 in the text.