Math 5615H **Midterm Exam 1**

Posted: 10/9; some clarifications: 10/15; due Friday, 10/16, 1:10 p.m.

The problem set is due at the **end of the class time on Friday**, there will be no class meeting on that Friday. Use that time to finish and upload your exam on Canvas. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Rules: **Unlike working on your homework, no study groups or cooperation when doing the exam, no asking questions on internet forums, etc.! You may use any textbooks and internet sources, but just copying arguments you might occasionally find will not gain any credit and will be regarded as plagiarism. You have to present all solutions in your own words.**

Regarding justifying your solutions: You may use any statement stated in class or in our textbook, baby Rudin, or stated in the homework, unless it makes your solution ridiculous, such as "Stated in class." You may also use one exam problem in your solution of another exam problem. You may use whatever theorems of algebra you wish to use.

You should also write on your paper the following *honor pledge*: "I pledge my honor that I have not violated the Honor Code during this examination" and sign your name under it.

Problem 1. Prove that the set of algebraic numbers, *i.e.*, the set of complex numbers that are roots of non-zero polynomials in one variable with rational coefficients, is countable.

Problem 2. Describe all the automorphisms of the field $\mathbb{C}$ of complex numbers that fix $\mathbb{R}$ pointwise, *i.e.*, $\sigma : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ such that $\sigma(x) = x$ for all $x \in \mathbb{R}$.

Problem 3. If U is an open set of $\mathbb{R}$, then there is an at most countable collection of disjoint open intervals I_n, $n = 1, 2, \dots$ such that $U = \bigcup_{n\geq 1} I_n$.

Problem 4 (Ashmita Sarma)**.** Show that the closed unit ball $\{x \in \mathbb{R}^\infty \mid d(x, 0) \leq 1\}$ in $\mathbb{R}^\infty$ is closed and bounded but not compact. Here $\mathbb{R}^\infty := \bigcup_{n\geq 1} \mathbb{R}^n$, where $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. The distance function $d(x, y)$ for $x, y \in \mathbb{R}^\infty$ is defined as the distance in $\mathbb{R}^n$ with n large enough so that both x and y are in $\mathbb{R}^n$. Assume that the distance function is independent of the choice of n and defines the structure of a metric space.

Problem 5. Prove that if a subset $K \subset X$ of a metric space X has the *Bolzano-Weierstrass property*, *i.e.*, every infinite subset of K has a limit point (cluster point) in K, then K is compact. *Hint*: You may assume the homework problem on the existence of a(n at most) countable base of K and show that any open cover of K has an at most countable subcover, say $\{U_n \mid n \in \mathbb{N}\}$. Show that this cover must actually be finite by contradiction: if no finite subcollection of $\{U_n\}$ covers K, then each of the nested sequence of sets $F_n := K \setminus (U_1 \cup \cdots \cup U_n)$ must be nonempty, while $\bigcap_{n\geq 1} F_n$ must be empty. Take an infinite set E which contains a point from each F_n, consider a cluster point, and get a contradiction.

Problem 6. Recall that we used ternary representation of real numbers in the closed interval $[0, 1]$ with the convention that infinite tails of 2's were allowed only in the following cases: we were using the tail $\dots 0\overline{222}\dots$ instead of the tail $\dots 1\overline{000}\dots$ A ternary (base-3) expansion

$$0.b_1 b_2 b_3 \dots \text{ with } b_i = 0,\ 1, \text{ or } 2 \tag{1}$$

represents the number

$$\sup\{\sum_{j=1}^{n} b_j/3^j \mid n \in \mathbb{N}\}.$$

Show that the Cantor set C consists of all the numbers in $[0, 1]$ whose ternary expansion as above has only 0's and 2's:

$$C = \{x = 0.b_1 b_2 b_3 \dots \in [0, 1] \mid b_i = 0 \text{ or } 2\}.$$

Hint: Analyze the excluded sets D_1, D_2, ... using ternary expansions (1): describe all the ternary expansions for numbers that lie in D_1, then for numbers that lie in D_2, etc.

Problem 7. Show that intervals in $\mathbb{R}$ (defined as $[a, b]$, (a, b), $[a, b)$, $(a, b]$ for $a, b \in \mathbb{R} \cup \{\pm\infty\}$) are exactly those subsets I of $\mathbb{R}$ which contain all their intermediate points, *i.e.*, $\forall x, y, z \in \mathbb{R}$ such that $x < y < z$ and $x, z \in I$, we have $y \in I$. ("Exactly those" is an "if and only if" statement. It means show that the intervals contain all their intermediate points and every set which contains all its intermediate points must be an interval.)