Math 5615H **Solutions to Midterm Exam 1**
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Rules: **Unlike working on your homework, no study groups or cooperation when doing the exam, no asking questions on internet forums, etc.! You may use any textbooks and internet sources, but just copying arguments you might occasionally find will not gain any credit and will be regarded as plagiarism. You have to present all solutions in your own words.**

Regarding justifying your solutions: You may use any statement stated in class or in our textbook, baby Rudin, or stated in the homework, unless it makes your solution ridiculous, such as "Stated in class." You may also use one exam problem in your solution of another exam problem. You may use whatever theorems of algebra you wish to use.

You should also write on your paper the following *honor pledge*: "I pledge my honor that I have not violated the Honor Code during this examination" and sign your name under it.

Problem 1. Prove that the set of algebraic numbers, *i.e.*, the set of complex numbers that are roots of non-zero polynomials in one variable with rational coefficients, is countable.

Solution. The set $P_n = \{a_0 + a_1 z + \cdots + a_n z^n\}$ of polynomials of degree $\le n$ with rational coefficients a_i is countable, being equivalent to $\mathbb{Q}^{n+1} = \{(a_0, a_1, \dots, a_n)\}$, which is countable as a Cartesian product of countable sets. The set P of all polynomials is the union of P_n's over $n \in \mathbb{N}$, a countable union of countable sets, and is thereby countable. If we remove the zero polynomial from it, the remaining set is an infinite subset of the countable set P and is therefore countable. The set of algebraic numbers, usually denoted $\bar{\mathbb{Q}}$, is the set of roots of nonzero polynomials with rational coefficients:

$$\bar{\mathbb{Q}} = \bigcup_{p \in P \setminus \{0\}} \{z \in \mathbb{C} \mid p(z) = 0\}.$$

Since for each $p \in P$ the set $\{z \in \mathbb{C} \mid p(z) = 0\}$ of its roots is finite, we get a countable union of finite sets. Such a union has to be countable, unless it is finite, which is obviously not the case, as every rational is an algebraic number.

Problem 2. Describe all the automorphisms of the field $\mathbb{C}$ of complex numbers that fix $\mathbb{R}$ pointwise, *i.e.*, $\sigma : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ such that $\sigma(x) = x$ for all $x \in \mathbb{R}$.

Solution. Let σ be such an automorphism. Since σ respects addition and multiplication and fixes the real numbers, for a complex number $a + bi$, we have $\sigma(a + bi) = a + b\sigma(i)$. Thus, σ is determined by $\sigma(i)$. But $-1 = \sigma(-1) = \sigma(i^2) = (\sigma(i))^2$ and $\sigma(i)$ must square to -1, that is to say, must be a solution of the equation $z^2 + 1 = 0$. This equation has two solutions i and $-i$. Thus, $\sigma(i) = i$ or $\sigma(i) = -i$. In the first case, $\sigma = \mathrm{id}$, because $\sigma(a + bi) = a + bi$. In the second case, $\sigma(a + bi) = a - bi$, which is known as complex conjugation.

So far, we know that if σ is an automorphism, it must act on complex numbers as either identity, or complex conjugation. If these two are indeed automorphisms of $\mathbb{C}$ fixing each real number, then these two maps σ will provide a complete answer. The identity $\mathrm{id}(z) = z$ if obviously an automorphism, whereas complex conjugation $\sigma(z) = \bar{z}$ is also an automorphism, given that it takes 0 to 0, 1 to 1, and $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.

Problem 3. If U is an open set of $\mathbb{R}$, then there is an at most countable collection of disjoint open intervals I_n, $n = 1, 2, \dots$ such that $U = \bigcup_{n \geq 1} I_n$.

Solution. Given an open set U, $\mathbb{Q} \cap U \subset \mathbb{Q}$ is countable, unless $U = \varnothing$. In the latter case, the result is trivially true. So, assume $U \neq \varnothing$. Let us enumerate the rationals contained in it: $r_1, r_2, \dots, r_n, \dots$. For each $n \geq 1$, take the "largest" open interval $I_n \subset U$ containing r_n. More precisely, define

$$I_n := (a_n, b_n),$$

where

$$\begin{aligned} a_n &= \inf\{a \mid \exists b : r_n \in (a, b) \subset U\}, \\ b_n &= \sup\{b \mid \exists a : r_n \in (a, b) \subset U\}. \end{aligned}$$

For every real $x \in U$, there is an open interval (a', b') such that $x \in (a', b') \subset U$, because U is open. Therefore the above sets whose infimum and supremum we have taken are nonempty. There is nothing wrong if $a_n = -\infty$ or $b_n = \infty$: (a_n, b_n) would still be an open interval. Therefore, these infimum and supremum exist in $\mathbb{R} \cup \{\pm\infty\}$ for each n.

For the same reason, every real $x \in U$ will be in one of those intervals I_n, because (a', b') will contain at least one of the rationals r_n, given that they are dense in $\mathbb{R}$, and $x \in (a', b') \subset (a_n, b_n)$. Thus,

$$\bigcup_{n \geq 1} I_n = U.$$

By construction, two intervals I_m and I_n will have to coincide if they happen to intersect. Thus, if we drop the repeating terms in $\bigcup_{n \geq 1} I_n$, we get an at most countable union of disjoint intervals with the same property: $\bigcup_{n \geq 1} I_n = U$.

Problem 4 (Ashmita Sarma)**.** Show that the closed unit ball $\{x \in \mathbb{R}^\infty \mid d(x,0) \le 1\}$ in $\mathbb{R}^\infty$ is closed and bounded but not compact. Here $\mathbb{R}^\infty := \bigcup_{n\ge 1} \mathbb{R}^n$, where $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is given by $(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0)$. The distance function $d(x,y)$ for $x,y \in \mathbb{R}^\infty$ is defined as the distance in $\mathbb{R}^n$ with n large enough so that both x and y are in $\mathbb{R}^n$. Assume that the distance function is independent of the choice of n and defines the structure of a metric space.

Solution. The closed unit ball $\bar{B}$ in $\mathbb{R}^\infty$ is closed, because its complement is open. Indeed, every point $x \in \mathbb{R}^\infty \setminus \bar{B}$ will have an open ball B' about it such that $B' \cap \bar{B} = \varnothing$: just take the open ball $B' = B_\delta(x)$ or radius $\delta = d(x,0) - 1$; then for each $y \in B'$, we have $d(0,y) + d(y,x) \ge d(0,x)$ by the triangle inequality, whence $d(0,y) \ge d(0,x) - d(y,x) > d(0,x) - \delta = 1$, meaning $y \notin \bar{B}$.

The closed unit ball is bounded, because it is contained in a closed ball, namely itself.

If $\bar{B}$ were compact, it would satisfy the Bolzano-Weierstrass (BW) property, that is to say, each infinite subset in $\bar{B}$ would have a cluster point in $\bar{B}$. However, the collection of points

$$x_n := (0,0,\dots,0,1,0,0,\dots), \qquad 1 \text{ in the } n\text{th component}, \ n \ge 1,$$

in $\bar{B}$ will not have a cluster point not only in $\bar{B}$ but in $\mathbb{R}^\infty$. This is because infinitely many of these points will have to sit in an open ball of radius $1/2$ about the cluster point and, therefore, be at distance strictly less than 1 to each other, whereas $d(x_m, x_n) = 1$ as long as $m \ne n$.

Problem 5. Prove that if a subset $K \subset X$ of a metric space X has the *Bolzano-Weierstrass property*, *i.e.*, every infinite subset of K has a limit point (cluster point) in K, then K is compact. *Hint*: You may assume the homework problem on the existence of a(n at most) countable base of K and show that any open cover of K has an at most countable subcover, say $\{U_n \mid n \in \mathbb{N}\}$. Show that this cover must actually be finite by contradiction: if no finite subcollection of $\{U_n\}$ covers K, then each of the nested sequence of sets $F_n := K \setminus (U_1 \cup \dots \cup U_n)$ must be nonempty, while $\bigcap_{n\ge 1} F_n$ must be empty. Take an infinite set E which contains a point from each F_n, consider a cluster point, and get a contradiction.

Solution. Let $\{O_n \mid n \in J\}$, where $J = J_N = \{1,2,\dots,N\}$ or $J = \mathbb{N}$, be a countable base of K guaranteed by the homework problem. Given an open cover $\{V_\alpha \mid \alpha \in A\}$ of K, we can choose one $\alpha = \alpha(n) \in A$ for every $n \in J$ in such a way that $O_n \subset V_{\alpha(n)}$. (If for a given n, there is no such $\alpha(n)$, we will skip that n.) Since $\{O_n\}$ is a countable base for K, every point of K will be in at least one of these O_n's and, therefore, in at least one $V_{\alpha(n)}$. Now, the set of $\alpha(n)$'s for $n \in J'$, where $J' := J$ less the n's we skipped above, is the image of a subset of J, which is at most countable. Therefore, the set of $\alpha(n)$'s must be at most countable, too. This way, we get an at most countable subcover $\{U_n := V_{\alpha(n)} \mid n \in J'\}$ of the given open cover $\{V_\alpha\}$. If the subcover is finite, we are done. Let us assume it is countable: $\{U_n \mid n \in \mathbb{N}\}$.

Suppose no finite subcover of $\{U_n\}$ covers K for a contradiction. Then each set $F_n := K \setminus (U_1 \cup \cdots \cup U_n)$ must be nonempty, while $\bigcap_{n\geq 1} F_n = K \setminus \bigcup_{n\geq 1} U_n$ must be empty. Note that $F_{n+1} \subset F_n$ for each n, *i.e.*, the sets are nested. Form a set E by taking a point $x_n \in F_n$ for each n. This set cannot be finite, because otherwise one of the points x_n will be in $\bigcap_{n\geq 1} F_n$, which is empty. By the BW assumption, the set E has a cluster point x in K. Since $\{U_n\}$ covers K, there is some n such that $x \in U_n$. There is an open ball in X centered at x and contained in U_n, because U_n is open. The set U_n does not intersect $F_n = K \setminus (U_1 \cup \cdots \cup U_n)$ and all the F_m's for $m \geq n$, as they are contained in F_n. Therefore, the open ball centered at x does not intersect these F_m's and hence cannot contain points x_m for any $m \geq n$. This contradicts the fact that x is a cluster point of the set E of all x_k's.

Problem 6. Recall that we used ternary representation of real numbers in the closed interval $[0, 1]$ with the convention that infinite tails of 2's were allowed only in the following cases: we were using the tail $\dots 0\overline{222}\dots$ instead of the tail $\dots 1\overline{000}\dots$ A ternary (base-3) expansion

$$0.b_1b_2b_3\dots \text{ with } b_i = 0,\ 1, \text{ or } 2 \tag{1}$$

represents the number

$$\sup\{\sum_{j=1}^{n} b_j/3^j \mid n \in \mathbb{N}\}.$$

Show that the Cantor set C consists of all the numbers in $[0, 1]$ whose ternary expansion as above has only 0's and 2's:

$$C = \{x = 0.b_1b_2b_3\dots \in [0, 1] \mid b_i = 0 \text{ or } 2\}.$$

Hint: Analyze the excluded sets D_1, D_2, ... using ternary expansions (1): describe all the ternary expansions for numbers that lie in D_1, then for numbers that lie in D_2, etc.

Solution. The set $D_1 = (1/3, 2/3)$ consists precisely of those reals x whose ternary expansion (with our current assumption on the tails) is

$$x = 0.1b_2b_3\ldots,$$

because $1/3 = 0.1\overline{000}\ldots$ and $2/3 = 0.2\overline{000}\ldots$. Since $1/3$ itself is replaced with $0.0\overline{222}\ldots$, points x in $C_1 = [0,1] \setminus D_1$ will be exactly those which have $b_1 \neq 1$.

The set D_2 is the union $(1/9, 2/9) \cup (7/9, 8/9)$ of the two middle thirds of the remaining two intervals $[0, 1/3]$ and $[2/3, 1]$ in $C_1 = [0,1] \setminus D_1$. For numbers $x \in [0, 1/3]$, $b_1 = 0$ and, for $x \in [2/3, 1]$, $b_1 = 2$. Which third of each of these two closed intervals x belongs to is characterized by the second ternary digit, called *trit*, b_2: $b_2 = 1$ precisely when x is in the open middle third of either closed interval. Thus, $x \in D_2$ iff $b_1 \neq 1$ and $b_2 = 1$. Note that the elements of $C_2 = C_1 \setminus D_2$ are precisely those x for which $b_1 \neq 1$ and $b_2 \neq 1$.

Inductively, if we know that $x \in D_n$ are characterized by $b_1, b_2, \ldots, b_{n-1} \neq 1$ and $b_n = 1$ and the elements of $C_n = C_{n-1} \setminus D_n$ are characterized precisely by $b_1, \ldots, b_n \neq 1$, then the elements of the middle thirds of the closed intervals comprising C_n will be those $x \in C_n$ for which $b_{n+1} = 1$. These are exactly the elements of D_{n+1}. Whence, the elements of $C_{n+1} = C_n \setminus D_{n+1}$ are exactly those for which $b_1, \ldots, b_n, b_{n+1} \neq 1$.

Thus, points of $C = \bigcap_{n=0}^{\infty} C_n$ are precisely those x for which all the trits $b_n \neq 1$.

Problem 7. Show that intervals in $\mathbb{R}$ (defined as $[a,b]$, (a,b), $[a,b)$, $(a,b]$ for $a, b \in \mathbb{R} \cup \{\pm\infty\}$) are exactly those subsets I of $\mathbb{R}$ which contain all their intermediate points, *i.e.*, $\forall x, y, z \in \mathbb{R}$ such that $x < y < z$ and $x, z \in I$, we have $y \in I$. ("Exactly those" is an "if and only if" statement. It means show that the intervals contain all their intermediate points and every set which contains all its intermediate points must be an interval.)

Solution. (1) If I is an interval $\langle a, b\rangle$, where $\langle$ is (or [and $\rangle$ is) or], and $a, b \in \mathbb{R} \cup \{\pm\infty\}$, let us show I contains all its intermediate points. Indeed, whenever for $x, y, z \in \mathbb{R}$, $x < y < z$ with $x, y \in I$, we will at least have $a \leq x < z \leq b$. This implies that $a < y < b$, *i.e.*, $y \in (a,b) \subset \langle a, b\rangle$.

(2) Suppose $I \subset \mathbb{R}$ contains all its intermediate points. Take $a := \inf I$ and $b := \sup I$. Let us allow them to be $\pm\infty$. Then these infimum and supremum will always exist, unless $I = \varnothing$, in which case $I = \varnothing = [1, 0]$ is trivially an interval. Let us first show that $(a, b) \subset I$. If $y \in (a, b)$, then there exists $x \in I$ such that $a \leq x < y$. Otherwise, y would be a greater lower bound of I than a. Similarly, there exists

$z \in I$ such that $y < z \le b$. Since I contains all its intermediate points, y must also be in I. Therefore, we see that $(a, b) \subset I$.

By construction, I may possibly differ from (a, b) by containing one or both points a and b. Thus, $I = \langle a, b \rangle$ with conventions as in Part (1) of this solution.