Math 5615H **Homework 10**

Posted: 11/14; Updated 11/19; due: Friday, 11/21/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 5: pages 103-109.

For this homework, you may assume that $(\sqrt{x})' = 1/(2\sqrt{x})$, $\sin' x = \cos x$ and similar computations known to you from calculus.

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

At which points is the function differentiable?

Problem 2. Give the critique of (*i.e.*, find a gap in) the following supposed "proof" of the chain rule:

$$\begin{aligned} \lim_{t\to x} \frac{g(f(t)) - g(f(x))}{t - x} &= \lim_{t\to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \frac{f(t) - f(x)}{t - x} \\ &= \left(\lim_{t\to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \right) \left(\lim_{t\to x} \frac{f(t) - f(x)}{t - x} \right) \\ &= g'(f(x))f'(x). \end{aligned}$$

Problem 3. Consider the function

$$f(x) = \begin{cases} 2x^2 \sin(1/x) + x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that f has a positive derivative at $x = 0$ but is not monotonically increasing in any neighborhood of $x = 0$.

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function whose derivative exists at each point and is bounded. Show that f is uniformly continuous.

Problem 5.
(1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{|x|\to+\infty} f(x) = 0$. Prove that f is uniformly continuous.
(2) Find a bounded function $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at every point and uniformly continuous, but f' is not bounded. *Hint*: Make use of Part (1). Note that the derivative must oscillate between large positive and negative values as $|x| \to \infty$, because if, say, f' just grows unboundedly with $x \to +\infty$, then it should force f to do the same. For instance, try to see why $\sin x/x$ does not work and cook something based on that.

Problem 6.
(1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at $x = 0$.
(2) Find a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at one point and not continuous at any other point.

Problem 7. Suppose that f is differentiable at each point of (a, b) and its derivative is never 0. Prove that f is strictly increasing or strictly decreasing on the interval. (Note that f' is not assumed to be continuous.)

Problem 8. Suppose f is a real-valued function on $(0, +\infty)$ with the properties:
(1) $f(xy) = f(x) + f(y)$ for all positive x and y;
(2) $f'(1)$ exists and equals 1.

Prove that $f(1) = 0$ and $f'(x)$ exists and equals $1/x$ for all $x > 0$. *Hint*: For the second statement, do $x + h = x(1 + h/x)$.