Math 5615H **Homework 2**

Posted: 09/13, modified: 09/17; due: Friday, 09/19/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 1 from the Extended Real Number System through Euclidean Spaces. Skim the Appendix to Chapter 1, if you have not done so in the previous week. Chapter 2: Finite, Countable, and Uncountable Sets.

Problems:

1. Prove that there is no way to introduce an order on $\mathbb{C}$ so that it becomes an ordered field. *Hint*: Shall we have $i > 0$ or $i < 0$?.

2. Show that the field of complex numbers is not isomorphic to the field of real numbers. (An *isomorphism* between fields is a bijection that preserves addition and multiplication. We looked at *order isomoprhisms* between ordered fields, and an isomorphism is a similar notion.)

3. Suppose z is a nonzero complex number. Show that there is a unique pair $r \in \mathbb{R}$, $w \in \mathbb{C}$ such that $r > 0$, $|w| = 1$, and $z = rw$.

4. Define e^{it} for real t as $\cos t + i \sin t$. Show that $|e^{it}| = 1$ and $e^{is+it} = e^{is}e^{it}$. Show that every nonzero complex number can be factored into re^{it} with $r, t \in \mathbb{R}$ and $r > 0$. This is called *polar decomposition*.

5. Show, using polar decomposition, that for any $n \in \mathbb{N}$, there are exactly n solutions to the equation $z^n = 1$.

6. Show that any automorphism of $\mathbb{R}$ is trivial, *i.e.*, is the identity. *Hint*: First, show that the rationals must be fixed by an automorphism. Then show that an automorphism must preserve the order of the real numbers. (Surprise: algebra enforces analysis!)

7. Show that the taxicab metric in $\mathbb{R}^k$ is positive definite, symmetric, and satisfies the triangle inequality.

8. Show that there can be no bijection between a finite set and a proper subset of it. *Hint*: Use the pigeonhole principle and interpret it as the fact that a map from J_m to J_n with $m > n$ cannot be injective.

9. Show that the set of polynomials with integral coefficients is countable. *Hint*: Use the idea of "counting" the elements of $\mathbb{Q}$ we had in class.

10. If A is an infinite set, then A has a countable subset.

11. If A is a set (including the empty set), then there is no bijection between A and the set $P(A)$ of all subsets of A. *Hint*: Use the idea of the proof of Cantor's theorem we had in class, *i.e.*, assume there is a bijection $f : A \to P(A)$ and then construct a subset of A which could not be $f(a)$ for any $a \in A$.

12. (Challenge problem – more like an independent project, need to know/learn more than we know officially, such as groups and some field theory; okay to skip, not for credit, i.e., does not carry any points). Describe the automorphism group $\mathrm{Aut}(\mathbb{C})$.