Math 5615H **Homework 6**

Posted: 10/17 (first version), 10/21 (typos corrected); due: Friday, 10/24/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 3: pages 55-65.

Problem 1. If $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a metric space, show that for any $\epsilon > 0$ there exists a subsequence $\{p_{n_i}\}_{i\in\mathbb{N}}$ so that $d(p_{n_i}, p_{n_{i+1}}) < \epsilon/2^{i+1}$.

Problem 2. Show that any sequence $\{s_n\}$ in $\mathbb{R}$ has a monotonic subsequence. *Hint*: Start with supposing that it does not have a monotonically increasing subsequence.

Problem 3. Show that every bounded sequence of real numbers has a convergent subsequence.

Problem 4. Consider a geometric series $\sum_{n=0}^{\infty} z^n$ defined by a complex number z. Show that if $|z| < 1$, then the series converges, and that if $|z| > 1$, then the series diverges. You can use anything through page 63 in the text, including limits of sequences of real numbers, such as those given on page 57.

Problem 5. Show that for a sequence $\{p_n\}$ of real numbers,

$$\limsup_{n\to\infty} p_n = \lim_{n\to\infty} \sup_{m\geq n}\{p_m\}$$

and

$$\liminf_{n\to\infty} p_n = \lim_{n\to\infty} \inf_{m\geq n}\{p_m\}.$$

Hint: Of course, you should prove only one of these statements and turn the other one into the first one by a one-symbol algebraic trick.

Problem 6. Show that for a sequence $\{p_n\}$ of real numbers, $\limsup_{n\to\infty} p_n < +\infty$ iff $\{p_n\}$ is bounded above.

Problem 7. Let z be a complex number, $|z| < 1$. Prove that $\lim_{n\to\infty} z^n = 0$, using the ϵ-N definition of a limit. *Hint*: Prove and use the inequality $(1+h)^n \geq 1 + nh$ for any positive h and natural n. See also the (very sketchy) proof of Theorem 3.20 (d).

Problem 8. Suppose that $\{s_n\}$ is the sequence of partial sums of a series $\sum a_n$. Suppose that $\lim_{n\to\infty} a_n = 0$ and that $\lim_{n\to\infty} s_{2n}$ exists. Prove that the series converges.

Problem 9. Suppose that $a_1 \geq a_2 \geq a_3 \geq \dots$ and $\lim_{n\to\infty} a_n = 0$. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$. (This is easy if you know that $\lim_{n\to\infty} n a_n$ exists, but why should it exists?) *Hint*: Write what it means that 0 is not a limit of $\{n a_n\}$.

Problem 10. Let $x \in \mathbb{R}$, $x > 0$. Define $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (assume the series converges). Prove that $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$.