Math 5615H **Homework 8**

Posted: 11/01; due: Friday, 11/07/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 3: pages 75-78, Chapter 4: pages 83-89.

Problem 1. Determine whether the series converges or diverges:

(1) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}$; *Hint*: "Cancel" terms > 1.

(2) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$;

(3) $\sum_{n=1}^{\infty} (1-a)(1-\frac{a}{2})(1-\frac{a}{3}) \ldots (1-\frac{a}{n})$, $a > 0$; *Hint*: Use Raabe's test, which is Problem 6 from the previous homework.

(4) $1 + \frac{1}{3^2} - \frac{1}{2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4} + \cdots + \frac{1}{(4n+1)^2} + \frac{1}{(4n+3)^2} - \frac{1}{2n+2} + \ldots$

Problem 2. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Hint: If you are able to find the sum, it means you have an idea about "telescoping" series.

Problem 3. Determine the coefficients a_n of the power series whose sum is $(1-z)^{-2}$ for $|z| < 1$ by squaring $(1-z)^{-1}$.

Problem 4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ (reduced fraction)}, x \neq 0, \\ 0 & \text{if } x = 0 \text{ or } x \notin \mathbb{Q}. \end{cases}$$

Show that f is continuous at 0 and any irrational point, but not continuous at any nonzero rational point.

Problem 5. Describe all continuous functions $f : \mathbb{R} \to X$, where X is a discrete metric space.

Problem 6. Let S be a metric space and $q \in S$. Show that the distance function $d(p, q)$ is a continuous function of p.

Problem 7. Let E be a nonempty subset of a metric space S. Define the distance from a point $p \in S$ to the set E to be

$$d_E(p) = \inf\{d(p, q) \mid q \in E\}.$$

Prove that $d_E(p) = 0$ iff $p \in \overline{E}$, the closure of E. Prove that d_E is a continuous function on S.

Problem 8. Suppose that E is a subset of a metric space S that is not closed. Show that there is a continuous real-valued function on E that is not bounded.