MATH 5615H: HONORS ANALYSIS
SAMPLE FINAL EXAM (PART II)

INSTRUCTOR: SASHA VORONOV

You may not use a calculator, notes, books, etc. Only the exam paper, scratch paper, and a pencil or pen may be kept on your desk during the test. You must show all work.

Good luck!

Problem 1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$ and $||\mathbf{x}||$ denotes the Euclidean norm of $\mathbf{x}$, show that

$$\big|\, ||\mathbf{x}|| - ||\mathbf{y}|| \,\big| \le ||\mathbf{x} - \mathbf{y}||.$$

Problem 2. Is the set of finite subsets of $\mathbb{N}$ countable or uncountable? Prove your point.

Problem 3. Show that for any sequence $\{a_n\}$ of real numbers,

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

You may use any definition of the upper and lower limits.

Problem 4. Suppose that f is a bounded real-valued function on $\mathbb{R}$ with bounded and continuous first and second derivatives.

(1) Use Taylor's theorem around any fixed x to conclude that for all $h > 0$,

$$|f'(x)| \le \frac{2}{h} \sup_{x\in\mathbb{R}} |f(x)| + \frac{h}{2} \cdot \sup_{x\in\mathbb{R}} |f''(x)|.$$

(2) Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)|^2 \le 4 \sup_{x\in\mathbb{R}} |f(x)| \cdot \sup_{x\in\mathbb{R}} |f''(x)|$$

by choosing the best h in Part (1).

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