MATH 5615H: INTRODUCTION TO ANALYSIS I SAMPLE MIDTERM EXAM II

You may not use notes, books, etc. Only the exam paper, a pencil or pen may be kept on your desk during the test. Calculators are not allowed, either, but will not be needed. Ask me, and I will compute anything for you, if you need me to. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

Problem 1. Prove or disprove the following statements with a precise ε-δ argument for each.

(1) The function $f(x) = x$ is uniformly continuous for all real x.

(2) The function $g(x) = \sin x$ is uniformly continuous for all real x.

Solution. Answer: Uniformly continuous. Given an $\varepsilon > 0$, take $\delta = \min(\varepsilon/2, \pi/2, 1) > 0$. Then for any h such that $|h| < \delta$, we have

- $|\sin h| \le |h|$: $|\sin h|$ is the shortest distance from the point $|h|$ radians, which will be in the first quadrant, because $|h| < \pi/2$, on the unit circle to the x axis, whereas $|h|$ is the length of the path along the circle;
- $0 < 1 - \cos h = 2\sin^2(h/2) \le h^2/2 < |h|/2$, because $|h| < 1$.

Thus, $|\sin(x + h) - sinx| = |\sin x(\cos h - 1) - \sin h \cos x| \le |\sin x| \cdot |\cos h - 1| + |\sin h| \cdot |\sin x| \le 1 - \cos h + |\sin h| < |h|/2 + |h| < \varepsilon/4 + \varepsilon/2 < \varepsilon$. □

(3) The function $(f \cdot g)(x) = x \sin x$ is uniformly continuous for all real x.

Solution. Answer: Not uniformly continuous. Take $\varepsilon = 1$. Then for any given $\delta > 0$, take $\delta_1 = \min(\pi/2, \delta/2) > 0$ and $n \in \mathbb{N}$ such that $2\pi n \sin \delta_1 > 1$. Such n exists because we can apply the Archimedean principle to $2\pi \sin \delta_1$ which is > 0, given that we made sure that $0 < \delta_1 < \pi/2$. Then take $x = 2\pi n$ and $y = 2\pi n + \delta_1$. Then $|x - y| = \delta_1 < \delta$, but $|x \sin x - ysiny| = |y \sin y| = (2\pi n + \delta_1) \sin \delta_1 > 2\pi n \sin \delta_1 > 1$. □

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Problem 2. Suppose that f is a bounded real-valued function on $\mathbb{R}$ with bounded and continuous first and second derivatives.

(1) Use Taylor's theorem around any fixed x to conclude that for all $h > 0$,
$$|f'(x)| \leq \frac{2}{h} \sup_{x\in\mathbb{R}} |f(x)| + \frac{h}{2} \cdot \sup_{x\in\mathbb{R}} |f''(x)|.$$

(2) Prove that
$$\sup_{x\in\mathbb{R}} |f'(x)|^2 \leq 4 \sup_{x\in\mathbb{R}} |f(x)| \cdot \sup_{x\in\mathbb{R}} |f''(x)|$$
by choosing the best h in Part (1).

Problem 3. Prove that the improper integral
$$\int_0^\infty \frac{\cos(2x)}{x^{1/3}} dx$$
converges.

Solution. The integral is improper, because the integrand is not defined at 0 and the upper limit of integration is $+\infty$. Thus, by definition, the integral converges iff both $\int_0^1 \frac{\cos(2x)}{x^{1/3}} dx$ and $\int_1^\infty \frac{\cos(2x)}{x^{1/3}} dx$ converge. The first integral converges, because $\frac{|\cos(2x)|}{x^{1/3}} \geq \frac{1}{x^{1/3}}$ and $\int_0^1 \frac{1}{x^{1/3}} dx$ converges. The second integral converges, because of Dirichlet's test: $f(x) = \cos(2x)$ is continuous with $\int_1^b \cos(2x) dx = \sin(2b)/2 - \sin(2)/2$ bounded, $g(x) = x^{-1/3}$ is such that $g'(x) = -x^{4/3}/3 > 0$ is locally integrable on $[1, +\infty)$ and $\lim_{x\to+\infty} g(x) = 0$. $\square$

Problem 4. Suppose that $a_n > 0$, and $\sum_{n=1}^\infty a_n$ diverges. Prove that the following series must also diverge:
$$\sum_{n=1}^\infty \frac{a_n}{1 + a_n}.$$