Math 5616H **Homework 1**

Posted: 1/23; Updated: 1/24; Due: Friday, 1/30/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 6: Everything through 6.6, then 6.8, 6.12(a)-(d), 6.13(b), and 6.20-22.

Problem 1. Prove Theorem 6.12(b) for Riemann-integrable functions f_1 and f_2 on an interval $[a, b]$. Assume $\alpha(x) = x$.

Problem 2. Suppose that f is a continuous, real-valued function on an interval $[a, b]$. Prove that there exists $x \in [a, b]$ such that $\int_a^b f dx = f(x)(b - a)$.

Problem 3. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is integrable on $[0, 1]$. *Hint*: Try to use uniform continuity of $f(x)$ on an interval $[\varepsilon/4, 1]$ to find a partition P with $U(P, f) - L(P, f) < \varepsilon$.

Problem 4. Suppose that f is continuous and nonnegative on the interval $[a, b]$ and $\int_a^b f dx = 0$. Prove that $f \equiv 0$ on a $[a, b]$.

Problem 5. Let $f(1/n) = 1$ for $n \in \mathbb{N}$ and $f(x) = 0$ otherwise. Prove that f is integrable on $[0, 1]$.

Problem 6. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q > 0 \text{ (reduced rational fraction)}, \\ 0, & \text{if } x = 0 \text{ or irrational}. \end{cases}$$

Show that f is integrable on $[0, 1]$. FYI: This is an example of a function that is integrable on an interval but not differentiable at any point thereof.

Problem 7. Suppose that f is continuous, nonnegative, and monotonically increasing on $[0, \infty)$. Prove that

$$\int_0^x f(t)dt \leq x f(x)$$

for all $x \geq 0$.

Problem 8. Prove that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \left(\frac{1}{N + in} + \frac{1}{N - in} \right) = 2 \int_{-1}^{1} \frac{dt}{1 + t^2}.$$

Hint: Represent the left-hand side as a Riemann sum, which we will study on Wednesday.