Math 5616H **Homework 5**

Posted: 2/21; Updated: 2/22; Due: Friday, 2/27/2015

The problem set is due at the beginning of the class on Friday.

Reading: 7.10, Chapter 8: pp. 172–174, 178–181, 182–185.

Problem 1. Using differentiation, determine the coefficients a_n, $n \geq 0$, of the power series whose sum is $(1-z)^{-2}$ for $|z| < 1$.

Problem 2. Prove that, for any integer $k \geq 0$,

$$\sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}, \qquad |z| < 1.$$

Problem 3. Suppose that $f(z) = \sum_{n=1}^{\infty} a_n z^n$ has a radius of convergence $R > 0$, and suppose that $|z_0| = r < R$. Define

$$g(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n, \qquad |z - z_0| < R - r.$$

Prove that $g(z)$ is given by a convergent power series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

whose radius of convergence is at least $R - r$. *Hint*: Use the Binomial theorem to compute the coefficients b_n and then use the previous problem to estimate the lim sup formula for the radius of convergence of g using the radius of convergence R of f.

Problem 4. Determine the coefficients of the power series that defines a function with the following properties: $f''(z) = -f(z)$, $f(0) = 1$, $f'(0) = 0$.

Problem 5. Show that $e^{z_1} = e^{z_2}$, where $e^z := E(z)$, if and only if $z_1 - z_2 = 2\pi n i$ for some $n \in \mathbb{Z}$.

Problem 6. Extend $C(x)$ and $S(x)$ from (46) in the textbook to complex z:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Is it true that for each $w \in \mathbb{C}$ there is $z \in \mathbb{C}$ such that $\cos z = w$? Find all solutions when there are any.

Problem 7. Find ten other proofs of the Fundamental Theorem of Algebra. (No need to hand in)