Math 5616H **Homework 8**

Posted: 3/29; Updated: 4/1; Due: Friday, 4/3/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 11: Sections 1-8, 12.

Problem 1. Show that if we add one more property to the definition of a σ-ring $\mathfrak{M}$ on a set X in 11.1 of the text: $\mathfrak{M}$ contains the ambient set X, we will get a notion equivalent to the notion of a σ-algebra from class (a collection of subsets containing the empty set $\varnothing$ and closed under countable unions and complements).

Problem 2. Is the Cantor set a Borel set on $\mathbb{R}$?

Problem 3. Let X be a separable metric space and let $\mathfrak{M}$ be the σ-algebra generated by open balls in X. Show that $\mathfrak{M}$ contains all the open sets in X and all the closed sets. Describe some sets in an example $\mathfrak{M}$ that are neither open nor closed. The σ-algebra $\mathfrak{M}$ is called the σ-algebra of *Borel sets* in X. *Hint*: Show that every open set in X is a countable union of open balls.

Problem 4. Can a σ-algebra in a set X have cardinality $\aleph_0$? *Hint*: Show that if a σ-algebra is infinite, then it contains a countable collection of pairwise disjoint subsets. Deduce that the σ-algebra will also contain arbitrary unions of these dijoint subsets.

Problem 5. Show that if μ is a measure on a set X with a σ-algebra $\mathfrak{M}$ and there is a set $A \in \mathfrak{M}$ such that $0 < \mu(A) < \infty$, then $\mu(\varnothing) = 0$.

Problem 6. Show that the σ-algebra generated by the collection of open rectangles (called *intervals* in the text) is the same as the σ-algebra generated by the collection of half-open rectangles.

Problem 7. Show that the Lebesgue outer measure of a face $I_1 \times \cdots \times I_{i-1} \times \{a\} \times I_{i+1} \times \cdots \times I_n$ of a rectangle $I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ is zero. Here $I_1, \dots, I_n$ are open intervals in $\mathbb{R}$ and a is one of the endpoints of I_i.

Problem 8. Show that the restriction $\mu^*|_E$ of an outer measure μ^* on a set X to a subset $E \subset X$ defined by $\mu^*|_E(A) := \mu^*(E \cap A)$ for any $A \subset X$ is an outer measure on X. Show that any set that is measurable with respect to μ^* is measurable with respect to $\mu^*|_E$ (regardless of whether E is measurable).