# MATH 5616H INTRODUCTION TO ANALYSIS II
# SAMPLE FINAL EXAM: SOLUTIONS

You may not use notes, books, etc. Only the exam paper, a pencil or pen may be kept on your desk during the test. Calculators are not allowed, either, but will not be needed. Ask me, and I will compute anything for you, if you need me to. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

**Problem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $p > 0$, that is, $f(x + p) = f(x)$ for all $x$ and $p$ is the least positive number with this property. Prove that

$$\int_0^p f(x+y)dx = \int_0^p f(x)dx \qquad \text{for any } y \in \mathbb{R}.$$

*Solution.* For any real number $y$ and any integrable function $g : [a + y, b + y] \to \mathbb{R}$

$$\int_a^b g(x+y)dx = \int_{a+y}^{b+y} g(x)dx.$$

Of course we know this for continuous functions from Calculus, using a change of variables, but a quick comparison identifies the lower Riemann sums:

$$\begin{aligned} &L(g(x+y), a = x_0 \le x_1 \le \cdots \le x_n = b) \\ &= \sum_i \Big(\inf_{x\in[x_{i-1},x_i]} g(x+y)\Big)(x_i - x_{i-1}) \\ &= \sum_i \Big(\inf_{x\in[x_{i-1}+y,x_i+y]} g(x)\Big)(x_i + y - (x_{i-1}+y)) \\ &= L(g(x), a + y = x_0 + y \le \cdots \le x_n + y = b + y). \end{aligned}$$

Thus, the integrals, being the suprema of lower Riemann sums, must be equal.

Using this statement, we get

$$\int_0^p f(x+y)dx = \int_y^{p+y} f(x)dx = \int_y^{np} f(x)dx + \int_{np}^{p+y} f(x)dx,$$

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where $np$ is such that $y \leq np \leq p+y$. Since $f(x)$ takes the same values on $[np, p+y]$ and $[(n-1)p, y]$, we have $\int_{np}^{p+y} f(x)dx = \int_{(n-1)p}^{y} f(x)dx$. Thus $\int_0^p f(x+y)dx = \int_{(n-1)p}^{np} f(x)dx = \int_0^p f(x)dx$. □

**Problem 2.** (1) Find real numbers $a$ and $b$ such that the partial differential equation (PDE) (you do not need to know what it is, just the equation below)

$$u_t(t,x) = (k-1)u_x(t,x) + u_{xx}(t,x) - ku(t,x), \qquad k \in \mathbb{R},$$

turns into $w_t(t,x) = w_{xx}(t,x)$ after the substitution

$$u(t,x) = e^{ax+bt}w(t,x).$$

(2) Find an equation relating the derivatives (a.k.a. differentials) $Du(t,x)$ and $Dw(t,x)$. If there are derivatives of other functions involved, compute them.

*Solution.* (1) Using the product rule for partials, which may be thought of as derivatives of functions $\mathbb{R}$ to $\mathbb{R}$, we obtain

$$\begin{aligned} u_t &= e^{ax+bt}(bw + w_t), \\ u_x &= e^{ax+bt}(aw + w_x), \\ u_{xx} &= e^{ax+bt}(a^2w + 2aw_x + w_{xx}). \end{aligned}$$

Plug these in the PDE for $u$, and you will see that it is equivalent to the PDE for $w$ iff the coefficients by $w$ cancel: $b = (k-1)a + a^2 - k$. and the coefficients by $w_x$ cancel: $(k-1) + 2a = 0$, whence $a = (1-k)/2$ and $b = (1+k)^2/4$.

(2) By the product rule for functions $\mathbb{R}^2 \to \mathbb{R}$, we have $Du = wD(e^{ax+bt}) + e^{ax+bt}Dw = e^{ax+bt}(w \cdot (b,a) + Dw)$. □

**Problem 3.** Let $\{f_i \mid i \in I\}$ be a uniformly bounded set of Riemann integrable functions on $[a,b] \subset \mathbb{R}$. Define

$$F_i(x) := \int_a^x f_i(t)dt, \qquad a \leq x \leq b.$$

Show that the family $\{F_i\}$ contains a uniformly convergent subsequence.

*Solution.* Uniform boundedness for $f_i$'s means: there exists $M$ such that $|f_i(x)| \leq M$ for all $i$ and $x$. Therefore,

$$|F_i(x) - F_i(y)| = \left| \int_x^y f_i(t)dt \right| \leq M|x-y|,$$

which implies that the family $\{F_i\}$ is equicontinuous. It is also uniformly bounded, because $|F_i(x)| = |F_i(x) - F_i(a)| \leq M(x-a) \leq$

$M(b-a)$ for all $x \in [a,b]$. By the Arzelà-Ascoli theorem (every pointwise bounded, equicontinuous family of real-valued functions on a compact has a uniformly convergent subsequence), we get the result. $\square$

**Problem 4.** Suppose $f \in L^1([0,1])$, *i.e.*, $f : [0,1] \to [0,\infty]$ is an integrable function. Prove that $\lim_{\varepsilon \to 0+} \int_{[0,\varepsilon]} f dm = 0$.

*Solution.* By the sequential characterization of limits of functions, it suffices to show that for any monotone decreasing sequence $\{a_n\}$ such that $a_n \to 0+$, we have $\lim_{n\to\infty} \int_{[0,a_n]} f dm = 0$. Recall that $E \mapsto \nu(E) = \int_E f dm$ defines a measure on $[0,1]$. For $E_n = [0,a_n]$, we have $\nu(E_n) = \int_{E_n} f dm < \infty$, because $f \in L^1$, and $E_n \supset E_{n+1}$. By continuity of a measure from below, we get $\lim_{n\to\infty} \int_{E_n} f dm = \lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcap_n E_n\right) = \nu(\{0\}) = \int_{\{0\}} f dm = f(0) \cdot m(\{0\}) = f(0) \cdot 0 = 0$. $\square$

**Problem 5.** Prove that if $f$ is continuous on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f(x^n) dx = f(0).$$

*Solution.*

$$\int_0^1 f(x^n) dx = \int_0^c f(x^n) dx + \int_c^1 f(x^n) dx$$

for any $c \in (0,1)$.

Since $x^n \to 0$ uniformly on $[0,c]$ (as for $x \in [0,c]$ we have $0 \le x^n \le c^n \to 0$, which explains why we decided to split the integral in the first place), we claim that $f(x^n) \to f(0)$ also uniformly on $[0,c]$. Indeed, for each $\varepsilon > 0$ find $\delta > 0$ such that $|f(y) - f(0)| < \varepsilon$ for each $y : 0 \le y < \delta$. Then for this $\delta$ find an $N \in \mathbb{N}$ such that $0 \le x^n < \delta$ for all $x \in [0,c]$ and $n > N$. Then for such $n$ we have $|f(x^n) - f(0)| < \varepsilon$ for all $x \in [0,c]$.

Now, as $c \to 1-$, we will have $\int_0^c f(x^n) dx \to \int_0^c f(0) dx = f(0) \cdot c \to f(0)$ (passing to uniform limit inside Riemann integral). On the other hand, $\left| \int_c^1 f(x^n) dx \right| \le M(1-c) \to 0$ as $c \to 1-$, where $M$ is a bound for our continuous $f(y)$ on $[0,1]$. $\square$

**Problem 6.** Let $X$ and $Y$ be compact metric spaces and let $f(x,y) \in C(X \times Y)$ be a continuous real-valued function on $X \times Y$. Show that for every $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in C(X)$ and $h_1, \ldots, h_n \in C(Y)$ such that

$$\left| f(x,y) - \sum_{i=1}^n g_i(x) h_i(y) \right| < \varepsilon \quad \text{for all } (x,y) \in X \times Y.$$

*Solution.* Consider

$$\mathcal{A} := \{\text{functions on } X \times Y \text{ of the form} \sum_{i=1}^{n} g_i(x)h_i(y), \\ \text{as in the problem}\} \subset \mathcal{C}(X \times Y).$$

This is an algebra: obviously closed under addition, multiplication:

$$\left(\sum_{i=1}^{m} g_i(x)h_i(y)\right) \cdot \left(\sum_{j=1}^{n} s_j(x)t_j(y)\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} g_i(x)s_j(x)h_i(y)t_j(y),$$

and scalar multiplication by real numbers. It separates points. Indeed, for any $(x_1, y_1) \neq (x_2, y_2)$, we can assume WLOG that $x_1 \neq x_2$. Then we can take any continuous $g(x)$ such that $g(x_1) \neq g(x_2)$, such as the distance function $g(x) = d(x, x_1)$, and the function $g(x) \cdot 1 \in \mathcal{A}$ will separate $(x_1, y_1)$ and $(x_2, y_2)$. The algebra has the function $1 = 1 \cdot 1$ in it, so there is no point in $X \times Y$ at which all functions from $\mathcal{A}$ vanish. Thus, by the Stone theorem, $\mathcal{A}$ is uniformly dense in $\mathcal{C}(X \times Y)$, which rewrites exactly as the conclusion of the problem. □

**Problem 7.** Show that any measure is countably subadditive.

*Solution.* Let $(X, \mathfrak{M}, \mu)$ be our measure space. Given $A_n \in \mathfrak{M}$ for $n \in \mathbb{N}$, define $C_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$. Then $A_n$ does not intersect any of the $A_i$'s with $i < n$ and neither does $C_n$ with any of the $C_i$'s with $i < n$. Also, $\bigcup_n A_n = \coprod_n C_n$. By the countable additivity of $\mu$, we have $\mu\left(\bigcup_n A_n\right) = \mu\left(\coprod_n C_n\right) = \sum_n \mu(C_n) \leq \sum_n \mu(A_n)$, the last inequality being because $C_n \subset A_n$ for all $n$. □

**Problem 8.** Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ \frac{1}{\sqrt{d}} & \text{if } x \text{ is irrational and } x = 0.0\dots0d\dots, \end{cases}$$

where $d$ is the first nonzero digit in the decimal expansion of $x$. Prove that $f$ is measurable.

*Solution.* Note that the function $f$ is a simple function, *i.e.*, a linear combination of characteristic functions:

$$f = \sum_{d=1}^{9} \frac{1}{\sqrt{d}} \chi_{A_d},$$

where $A_d = \{x \in [0, 1] \mid x \notin \mathbb{Q}, d/10^n < x < (d+1)/10^n \text{ for some } n \geq 1\}$ for $1 \leq d \leq 8$ and $9/10^n < x < 1/10^{n-1}$ for $d = 9$. Each $A_d$ is Borel as an open set (a countable union of intervals) with $\mathbb{Q}$ removed, which is

Borel, being a countable union of points, each of which is closed. Hence each $A_d$ is Lebesgue measurable, and so is our simple function $f$. $\square$