MATH 8211: COMMUTATIVE AND HOMOLOGICAL ALGEBRA PROBLEM SET 3, DUE DECEMBER 12, 2003

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I encourage you to cooperate with each other on the homeworks.

Convention: all rings are commutative with an identity element $1 \neq 0$, all ring homomorphisms carry 1 to 1, and a subring shares the same identity element with the ring.

Problem 1. Find a ring A and a multiplicative set S such that the relation $(a, s) \sim (b, t) \iff at = bs$ is not an equivalence relation.

Problem 2. Let $M_i \subset M$ be submodules indexed by some set J, for which $M = \sum_{i \in J} M_i$, the sum of the submodules. Suppose that S is a multiplicative set, and $S^{-1}M_i = 0$ for all $i \in J$. Make an original discovery concerning $S^{-1}M$.

Problem 3. If $S = \{1, f, f^2, \dots\}$ is a multiplicative set of A, prove that $\operatorname{Spec}(A_f) \subset \operatorname{Spec} A$ is the complement of the closed set $\mathcal{V}(f)$.

Problem 4. Let $A = k[V] = k[X_1, \dots, X_n]/I(V)$ be the coordinate ring of a variety $V \subset k^n$ and $f \in A$. Prove that $A[1/f]$ is the coordinate ring of a variety $V_f \subset k^{n+1}$, which is in natural bijective correspondence with the open set $V \setminus V(f)$.

Problem 5. Exercise 2.2 of [E].

Problem 6. Exercise 2.8 of [E].

Problem 7. If M is an A-module, show that M can be identified with a certain subset of the sections of the surjection

$$\begin{aligned} \coprod_{P \in \operatorname{Spec} A} M_P &\to \operatorname{Spec} A, \\ m &\mapsto P \qquad \text{for } m \in M_P. \end{aligned}$$

If S is a multiplicative set, show that $S^{-1}M$ can be identified with a subset of partially defined sections, defined for P with $P \cap S = \emptyset$. By the way, M_P is called the *stalk of M over P*.

Problem 8. Give an example of a ring A and an ideal I which is not primary, but satisfies the condition $fg \in I \implies f^n \in I$ or $g^n \in I$ for some n. [Hint: that is, find a nonprimary ideal whose radical is prime.]

Problem 9 (Fitting's Lemma)**.** Let M be a Noetherian module and $\phi : M \to M$ a homomorphism. Prove that $\ker \phi^n \cap \operatorname{im} \phi^n = 0$ for some $n > 0$. [Hint: use our method of proving that every indecomposable ideal in a Noetherian ring is primary, when we were proving Noether's theorem on the existence of primary decomposition.]

Problem 10. Let $A = k[X, Y, Z]$ and I the ideal $(XY, X - YZ)$. Find a primary decomposition of I and determine the corresponding primes. [Hint: to guess the result, draw the variety $V(I)$. To prove it, note that the variety $X = YZ$ is

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isomorphic to the YZ-plane; consider $A \to k[Y, Z]$ sending X to YZ and restriction of primary ideals via this homomorphism.]

Problem 11. Let $A = k[X, Y, Z]/(XZ - Y^2)$ and $P = (x, y)$, and set $M = A/P^2$.

(1) Determine $\operatorname{Ass} M$. [Hint: use a primary decomposition of P^2 in A.]
(2) Find the elements of M annihilated by each assassin.
(3) Find a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} \cong A/P_i$, where $P_i \in \operatorname{Spec} A$, $i = 1, \dots, n$.

Problem 12. Exercise 3.1 of [E].

Problem 13. Exercise 3.3 of [E].

Problem 14. Using Nakayama's lemma, show that if (A, m) is a Noetherian local ring, then the maximal ideal m is principal, if and only if m/m^2 is one-dimensional over $k = A/m$. Show also that A is a DVR, if and only if A is Noetherian, local with $\operatorname{Spec} A = \{0, m\}$, and m/m^2 is one-dimensional over $k = A/m$.

Problem 15 (DVRs and nonsingular curves)**.** This problem says that A is a DVR, if and only if the plane curve $C := \{f = 0\} \subset k^2$ is nonsingular at $(0, 0)$.

Let k be an algebraically closed field and $f \in k[X, Y]$ an irreducible nonconstant polynomial of the form

$$f(X, Y) = l(X, Y) + g(X, Y)$$

with $l(X, Y) = aX + bY$ and $g \in (X, Y)^2$. Set $R = k[X, Y]/(f)$, $P = (X, Y)/(f)$, and $(A, m) = (R_P, m_P)$. Prove that A is a DVR, if and only if $l \neq 0$. [Hint: use Problem 14.]

That is it! Happy Thanksgiving!