LECTURE 16: DEFORMATION THEORY OF ASSOCIATIVE ALGEBRAS

ALEXANDER A. VORONOV

1. DEFORMATION THEORY OF ASSOCIATIVE ALGEBRAS

Let us start with a review of deformation theory of associative algebras. Let A be an associative algebra over a field k of characteristic zero.

Definition 1. A *formal deformation* of A is a $k[[t]]$-bilinear multiplication law $m_t : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]]$ on the space $A[[t]]$ of formal power series in a variable t with coefficients in A, satisfying the following properties:

$$m_t(a,b) = a \cdot b + m_1(a,b)t + m_2(a,b)t^2 + \cdots \qquad \text{for } a, b \in A,$$

where $a \cdot b$ is the original multiplication on A, and m_t is associative, which is equivalent to the equation

$$m_t(m_t(a,b),c) = m_t(a, m_t(b,c)) \qquad \text{for } a,b,c \in A.$$

Remark 2. Note that for a formal deformation m_t of a *commutative* algebra A, the bracket defined by the first-order part of the commutator,

$$
\begin{aligned}
\{a,b\} & := & \frac{1}{2t}(m_t(a,b) - m_t(b,a)) \mod t, \\
& = & \frac{1}{2}(m_1(a,b) - m_1(b,a))
\end{aligned}
$$

defines the structure of a Poisson algebra on A. (All the identities of a Poisson algebra follow from the fact that $A[[t]]$ with the product $m_t(a,b)$ and the bracket $\frac{1}{2}(m_t(a,b) - m_t(b,a))$ is a noncommutative Poisson algebra in an obvious sense, e.g., see [FGV95].) In physical terms, one can regard t as a quantum parameter, such as the Planck constant, the Poisson algebra A as the *quasi-classical limit* of the associative algebra $A[[t]]$, and the algebra $A[[t]]$ as a *deformation quantization* of the Poisson algebra A. The *deformation quantization problem* is the inverse problem: given a Poisson algebra A, find a formal deformation returning the original Poisson algebra structure on A in the quasi-classical limit.

The main tool in studying deformation theory is the *Hochschild complex*

$$0 \to C^0(A,A) \xrightarrow{d} \cdots \xrightarrow{d} C^n(A,A) \xrightarrow{d} C^{n+1}(A,A) \xrightarrow{d} \cdots,$$

where $C^n(A,A) := \mathrm{Hom}(A^{\otimes n}, A)$ is the space of *Hochschild n-cochains*, *i.e.*, the n-linear maps $f(a_1, \ldots, a_n)$ on A with values in A, and the *differential* d, $d^2 = 0$,

is defined as

$$(df)(a_1, \ldots, a_{n+1}) := a_1 f(a_2, \ldots, a_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_{n+1})$$
$$- (-1)^n f(a_1, \ldots, a_n) a_{n+1},$$

for $f \in C^n(A, A)$, $a_1, \ldots, a_{n+1} \in A$. The *Hochschild cohomology* is then the cohomology

$$H^\bullet(A, A) := \operatorname{Ker} d / \operatorname{Im} d$$

of this complex. The Hochschild complex admits a bracket

$$[,] : C^m(A, A) \otimes C^n(A, A) \to C^{m+n-1}(A, A),$$

called the *Gerstenhaber bracket, or the G-bracket*. The formula for this bracket, defined by M. Gerstenhaber [Ger63] is not really inspirational to me, in spite of all those years I have spent staring at it. A conceptual definition, due to J. Stasheff, is based on the following idea. The matter is that the Hochschild complex may be identified with the space of graded derivations of the tensor coalgebra: $T^c(A[1]) := \bigoplus_{n \geq 0} A[1]^{\otimes n}$, and the bracket is just the commutator of derivations [Sta93]. Here $A[1]$ denotes the graded vector space whose only nonzero graded component is A, placed in degree -1. In general, for a graded vector space $V = \bigoplus_n V^n$, $V[k]$ denotes *grading shift*, or what is known to topologists as *k-fold suspension*: it is a graded vector space $V[k]$ whose component of degree n is $V[k]^n := V^{k+n}$. Note that a derivation determined by a map $A[1]^{\otimes n} \to A[1]$ has degree $n - 1$, therefore, the bracket defines a differential graded (DG) Lie algebra structure on the Hochschild complex $C^\bullet(A, A)[1]$ with a shifted grading $\deg f = n - 1$ for $f \in C^n(A, A)$. Note that the (appropriately shifted) Hochschild cohomology $H^\bullet(A, A)[1]$ inherits the structure of a graded Lie algebra.

The importance of the bracket comes from the following tautological fact, which however may be regarded as the cornerstone of deformation theory.

Proposition 3. *A formal multiplication*

$$m_t(a, b) = m_0(a, b) + m_1(a, b)t + m_2(a, b)t^2 + \cdots, \qquad a, b \in A,$$

is associative, iff $[m_t, m_t] = 0$.

Proof. We need a little formula for the G-bracket of m_t with itself:

$$[m_t, m_t](a, b, c) = 2(m_t(m_t(a, b), c) - m_t(a, m_t(b, c))),$$

which would have been obvious, if we had used Gerstenhaber's formula to define the G-bracket. It is a good exercise to deduce it directly from the definition of the G-bracket as the commutator of derivations. The right-hand side of the formula contains the associativity equation, and we are done. □

Remark 4. Because the original multiplication $m_0(a, b) := a \cdot b$ is associative, the G-bracket square of it vanishes: $[m_0, m_0] = 0$. Therefore, the commutator with m_0 defines an inner differential on the Hochschild complex. This differential is in fact the Hochschild differential: another exercise is to verify that $df = [f, m_0]$.

Classical deformation theory was about the following "perturbative" results.

Corollary 5.   (1) *A formal multiplication*

$$m_t(a, b) = a \cdot b + m_1(a, b)t + m_2(a, b)t^2 + \cdots, \qquad a, b \in A,$$

is associative modulo t^2, iff $dm_1 = 0$. In this case m_1 defines a Hochschild cohomology class $m_1 \in H^2(A, A)$.

(2) *Suppose that a formal multiplication as above is associative modulo t^2. Then the existence of m_2 such that m_t is associative modulo t^3 is equivalent to the vanishing $[m_1, m_1] = 0$ in Hochschild cohomology $H^\bullet(A, A)$.*

Proof. Expand the equation $\frac{1}{2}[m_t, m_t] = 0$ in powers of t and collect terms by t^n for $n = 0$, 1, and 2. We will get the following.

$$t^0 : \qquad \frac{1}{2}[m_0, m_0] = 0,$$
$$t^1 : \qquad dm_1 = 0,$$
$$t^2 : \qquad dm_2 + \frac{1}{2}[m_1, m_1] = 0.$$

The first equation is always satisfied, because the original multiplication is associative, see Remark 4. The other two equations explain both statements of the corollary. □

References

[FGV95] M. Flato, M. Gerstenhaber, and A. A. Voronov, *Cohomology and deformation of Leibniz pairs*, Letters in Mathematical Physics **34** (1995), no. 1, 77–90.

[Ger63] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288.

[Sta93] J. Stasheff, *The intrinsic bracket on the deformation complex of an associative algebra*, J. Pure Appl. Algebra **89** (1993), 231–235.