LECTURE 17: DEFORMATION QUANTIZATION

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1. Deformation Quantization

Deformation quantization usually refers to a specific deformation quantization problem in a geometric/physical setting. The following theorem, which solves the deformation quantization problem posed by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [BFF+78], is a remarkable breakthrough in pure mathematics achieved by applying ideas motivated by Feynman diagrams.

Theorem 1 (M. Kontsevich [Kon97])**.** *Every Poisson manifold $(M, \{,\})$ may be* deformation quantized*, i.e., there exists a formal deformation quantization, see Remark 2 in Lecture 15,*

$$f \star g := m_t(f, g) = fg + m_1(f, g)t + m_2(f, g)t^2 + \cdots, \qquad f, g \in C^\infty(M),$$

of the Poisson algebra $A = C^\infty(M)$ of smooth functions, so that all the m_i's are local*, that is, bidifferential operators on M. According to our definition of deformation quantization, the star product must be associative and also recover the Poisson algebra of functions in the quasi-classical limit,* i.e., $(m_1(f, g) - m_1(g, f))/2 = \{f, g\}$.

Proof. We will only consider the case of $M = \mathbb{R}^d$ with an arbitrary Poisson structure, where the situation is already highly nontrivial. Globalization, which is done using a Fedosov-type connection, see [Kon97, CFT00], lies outside the main theme of these notes: no pattern in it has to do with graphs.

First, we will sketch Kontsevich's original proof, giving an explicit formula for the *star product* $f \star g$:

$$f \star g := \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in \mathcal{G}_{n,2}} W_\Gamma B_\Gamma(f, g). \tag{1.1}$$

which is explained in the this paragraph. The interior summation runs over the set $\mathcal{G}_{n,2}$ of directed graphs Γ of a certain type with vertices labeled $1, 2, \dots, n, \bar{1}, \bar{2}$. The set $\mathcal{G}_{n,2}$ of graphs consists of the graphs satisfying the following conditions. Each vertex of first type, *i.e.*, labeled 1, 2, ..., or n, has exactly two outgoing edges, labeled the first one and the second, and there are no other edges whatsoever. No edge may form a loop, *i.e.*, start and end at one and the same vertex. $B_\Gamma(f, g)$ is a bidifferential operator defined by an explicit formula [Kon97], which we will describe using an example.

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$$\text{For} \quad \Gamma = \begin{tikzcd} 1 \arrow[r, "j"] \arrow[d, "i"'] & 2 \arrow[dl, "k"'] \arrow[d, "l"] \\ \bar{1} & \bar{2} \end{tikzcd}$$

the corresponding bidifferential operator will be

$$B_\Gamma(f, g) := \alpha^{ij} \partial_j(\alpha^{kl}) \partial_k \partial_i(f) \partial_l(g),$$

where α^{ij} denotes the corresponding component of the Poisson tensor in a fixed coordinate system $(x_1, \dots, x_d)$ in $\mathbb{R}^d$ and we assume summation over the repeating indices. Finally, the coefficient W_Γ is given by another formula:

$$W_\Gamma := \frac{1}{n!(2\pi)^{2n}} \int_{C^+_{n,2}} \bigwedge_{\text{edges } r \to s \text{ of } \Gamma} d\phi(z_r, z_s),$$

where $C^+_{n,2}$ is the configuration space of n distinct points $z_1, \dots, z_n$ in the upper half-plane and two fixed points $z_{\bar{1}} = 0$ and $z_{\bar{2}} = 1$ on the real line; $\phi(z_r, z_s)$, r and s running over $\{1, 2, \dots, n, \bar{1}, \bar{2}\}$, is the directed angle at z_r between the hyperbolic line through z_r and ∞ and the hyperbolic line through z_r and z_s. The order in the wedge product is given by the lexicographic order of the vertices $\{1, \dots, n\}$ and the orders of the set of edges going out of the vertices. Kontsevich proves that this improper integral is absolutely convergent.

The associativity of the star product may be verified explicitly, see [Kon97]. However, I will sacrifice the rigor for the moral and give a more conceptual, physical explanation of the associativity, following A. Cattaneo and G. Felder [CF00]. Cattaneo and Felder define the star product as

$$(f \star g)(x) := \int_{\mathcal{P}_x} f(X(0)) g(X(1)) e^{iS(X,\eta)/t} DX D\eta. \tag{1.2}$$

This is a Feynman integral over the infinite dimensional "path" space, which is the following space of fields X and η on the upper half-plane H:

$$\mathcal{P}_x := \{X : \bar{H} \to \mathbb{R}^d, \eta \in \Omega^1(\bar{H}) \otimes \mathbb{R}^d \mid X(\infty) = x$$
$$\text{and } \eta \text{ vanishes on tangent vectors to the boundary}\}.$$

The function $S(X, \eta)$ is a certain action functional defining a Poisson sigma model on H, see [CF00]:

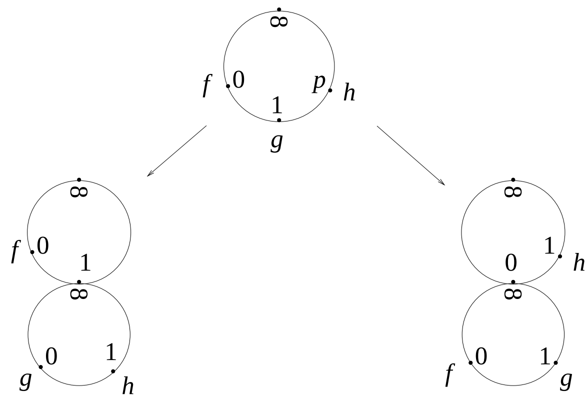
$$S = \int_H (\eta_{\mu i} \partial_\nu X^i + \frac{1}{2} \alpha^{ij}(X) \eta_{\mu i} \eta_{\nu j}) du^\mu du^\nu.$$

A rigorous definition of the Feynman integral would be the very formula (1.1). However, physics takes the opposite viewpoint and treats (1.1) as the saddle-point expansion of the integral (1.2) in parameter t obtained by formally applying the rules by analogy with the finite-dimensional case. The advantage of this approach is that the mystery of Kontsevich's formula (1.1) is now replaced by the mystery of Equation (1.2), which is not so mysterious to a physicist, for whom it represents a standard integral quantization formula. Another advantage is that it offers the following explanation of the associativity.

Consider the integral

$$\langle f, g, h\rangle_p(x) := \int_{\mathcal{P}_x} f(X(0))g(X(1))h(X(p))e^{iS(X,\eta)/t} DX D\eta,$$

where $p \in (1, \infty) \subset \mathbb{R} \subset \bar{H}$ is a fixed point on the real line between 1 and ∞ and $\mathcal{P}_x$ is as above in (1.2). This integral is independent of the choice of this point p, because the action S is diffeomorphism invariant and, roughly speaking, by integrating over all fields X and η, we take an average over all possible positions of p. Thus, the limits of $\langle f, g, h\rangle_p$ as $p \to 1$ and $p \to \infty$ will be the same. On the other hand, in the moduli space of configurations of four points $0, 1, p$, and ∞ on the boundary of H, these configurations will degenerate as follows:



which means that

$$\begin{aligned}\lim_{p\to 1}\langle f, g, h\rangle_p &= f \star (g \star h),\\ \lim_{p\to \infty}\langle f, g, h\rangle_p &= (f \star g) \star h,\end{aligned}$$

yielding the associativity.

In reality, things are more complicated than I have made them appear: the Feynman diagram expansion involves gauge fixing and renormalization, which is achieved by introducing ghosts and antighosts (and what not) and using the BV formalism, see [CF00]. The behavior of the Feynman integral with respect to the compactification of the configuration spaces is another issue suppressed in the above.

□

References

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[CFT00] A. S. Cattaneo, G. Felder, and L. Tomassini, *From local to global deformation quantization of Poisson manifolds*, Preprint, December 2000, `math.QA/0012228`.

[Kon97] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, Preprint, Institut des Hautes Études Scientifiques, September 1997, `math.QA/9709180`.