

LECTURE 5: TFT'S

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1. TFT's

Here we will consider a two-dimensional generalization of the graph PROP defining Lie bialgebras. Instead of graphs, 1d CW complexes, we will consider oriented topological surfaces, 2d topological manifolds, which you may think of as tubular neighborhoods of graphs in a three-space. Unlike the case of graphs, there will be no artificial equivalence relation imposed on the surfaces¹.

Consider the PROP of oriented topological Riemann surfaces with a finite number of boundary components labeled as m inputs and n outputs considered up to homeomorphism. (The boundaries are not allowed to intersect. Continuous parameterization of the boundary is still part of the data. However, it is known that there is a unique homeomorphism class of such connected surfaces a fixed genus g and the numbers m and n of inputs and outputs.) The PROP structure is again given by sewing the inputs of one given surface with the respective outputs another one, using the parameterizations of the boundary. The permutation group acts by relabeling, as usually.

Definition 1. A *Topological Field Theory (TFT)* is an algebra over the PROP of topological Riemann surfaces.

Again, here is an algebraic counterpart.

Definition 2. A *Frobenius algebra* structure on a vector space V over k means the structure of a commutative associative algebra with a unit and a nondegenerate symmetric bilinear form $\langle, \rangle : V \otimes V \rightarrow k$ which is invariant with respect to the multiplication: $\langle ab, c \rangle = \langle a, bc \rangle$. Here nondegenerate means that the canonical map $V \rightarrow V^*$ defined by the form is an isomorphism.

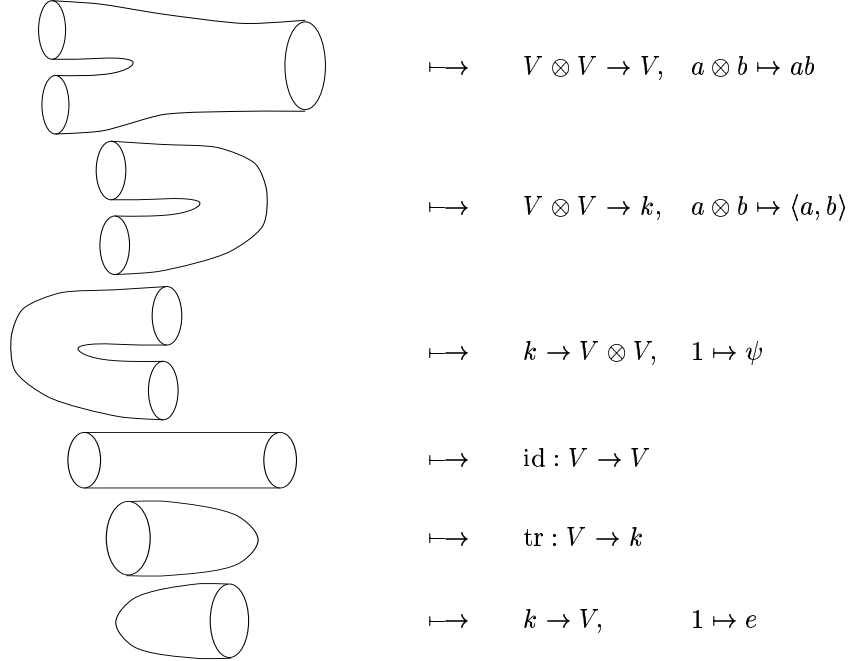
Theorem 3 (Folklore). *The structure of a TFT based on vector space V is equivalent to the structure of a finite-dimensional Frobenius algebra on it.*

Proof. Let V be the state space of a TFT. We would like to construct the structure of a Frobenius algebra on V . Consider the following Riemann surfaces and the

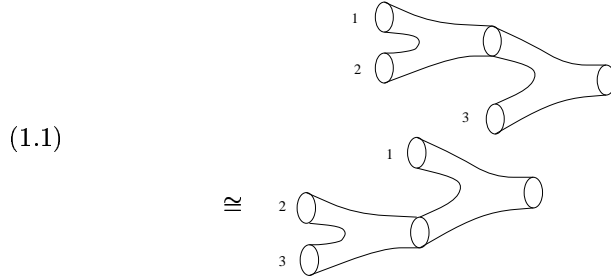
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¹As I have already remarked, the relations we placed on graphs are quite natural from the point of view of graph homology, which will be discussed later in the course

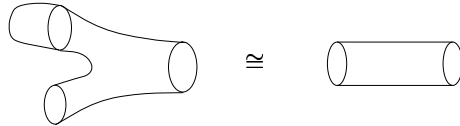
corresponding operators, which we denote ab , $\langle a, b \rangle$, etc.



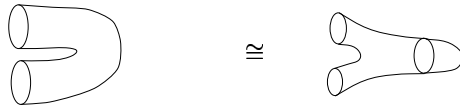
We claim that these operators define the structure of a Frobenius algebra on V . Indeed, the multiplication is commutative, because if we interchange labels at the legs of a pair of pants we will get a homeomorphic Riemann surface. Therefore, the corresponding operator $a \otimes b \mapsto ba$ will be equal to ab . Similarly, the associativity $(ab)c = a(bc)$ of multiplication is based on the fact that the following two surfaces are homeomorphic:



The property $ae = ea = a$ of the unit element comes from the homeomorphism



Thus, we see that V is a commutative associative unital algebra. Now, the following homeomorphism



proves the identity $\langle a, b \rangle = \text{tr}(ab)$, which, along with the associativity, implies $\langle ab, c \rangle = \langle a, bc \rangle$. The fact that the inner product $\langle \cdot, \cdot \rangle$ is nondegenerate follows from the homeomorphism



which implies that the composite mapping

$$V \xrightarrow{\text{id} \otimes \psi} V \otimes V \otimes V \xrightarrow{\langle \cdot, \cdot \rangle \otimes \text{id}} V, \\ v \mapsto \sum_{i=1}^n v \otimes u_i \otimes v_i \mapsto \sum_{i=1}^n \langle v, u_i \rangle v_i$$

is equal to $\text{id} : V \rightarrow V$. It takes a little linear-algebra exercise to see that this implies that the symmetric bilinear form is nondegenerate. This also implies that $\dim V < \infty$, because the v_i 's, $i = 1, \dots, n$, must span V . This completes the construction of the structure of a finite-dimensional Frobenius algebra on the state space V of a TFT.

Conversely, if we have a finite-dimensional Frobenius algebra V , we can define the structure of a TFT on the vector space V by (1) cutting a Riemann surface down into pairs of pants, cylinders, and caps; (2) defining the operators corresponding to those basic objects using the multiplication (or its linear dual), the identity map, and the unit element $e \in V$ (or the dual of the map $k \rightarrow V$, $1 \mapsto e$, as the trace functional), respectively; and (3) using the sewing axiom. The fact that the composite operator is independent of the way we cut down the surface follows from (1.1) and the associativity of multiplication. \square

Exercise 1. Is there a more elegant way (not using any elements of V) to show that $\dim V < \infty$?