LECTURE 6: OPERADS AND ALGEBRAS OVER THEM

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1. Operads

Now we are ready to deal with operads. Informally, an operad is the part $\mathrm{Mor}(n, 1)$, $n \geq 0$, of a PROP. Of course, given only the collection of morphisms $\mathrm{Mor}(n, 1)$, it is not clear how to compose them. The idea is to take the union of a m elements from $\mathrm{Mor}(n, 1)$ to be able to compose them with an element of $\mathrm{Mor}(m, 1)$. This leads to cumbersome notation and ugly axioms, compared to those of a PROP. However operads are in a sense more basic than the corresponding PROP's: the difference is similar to the difference between Lie algebras and the universal enveloping algebras.

Definition 1. An *operad* $\mathcal{O}$ is a collection of sets (vector spaces, complexes, topological spaces, manifolds, …, objects of a symmetric monoidal category) $\mathcal{O}(n)$, $n \geq 0$, with

(1) A composition law:
$$\gamma : \mathcal{O}(m) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m) \to \mathcal{O}(n_1 + \cdots + n_m).$$
(2) A right action of the symmetric group S_n on $\mathcal{O}(n)$.
(3) A unit $e \in \mathcal{O}(1)$.

such that the following properties are satisfied:

(1) The composition is associative, *i.e.*, the following diagram is commutative:
$$\begin{array}{ccc}
\left\{\begin{array}{l} \mathcal{O}(l) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_l) \\ \qquad \otimes \mathcal{O}(n_{11}) \otimes \cdots \otimes \mathcal{O}(n_{l,n_l}) \end{array}\right\} & \xrightarrow{\mathrm{id} \otimes \gamma^l} & \mathcal{O}(l) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_l) \\
\Big\downarrow{\scriptstyle \gamma \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \gamma} \\
\mathcal{O}(m) \otimes \mathcal{O}(n_{11}) \otimes \cdots \otimes \mathcal{O}(n_{m,n_m}) & \xrightarrow{\gamma} & \mathcal{O}(n)
\end{array},$$
where $m = \sum_i m_i$, $n_i = \sum_j n_{ij}$, and $n = \sum_i n_i$.

(2) The composition is equivariant with respect to the symmetric group actions: the groups S_m, S_{n_1}, …, S_{n_m} act on the left-hand side and map naturally to $S_{n_1 + \cdots + n_m}$, acting on the right-hand side.
(3) The unit e satisfies natural properties with respect to the composition: $\gamma(e; f) = f$ and $\gamma(f; e, \ldots, e) = f$ for each $f \in \mathcal{O}(k)$.

The notion of a *morphism of operads* is introduced naturally.

Remark 2. One can consider *non-Σ operads*, not assuming the action of the symmetric groups. Not requiring the existence of a unit e, we arrive at *nonunital operads*. Do not mix this up with operads with no $\mathcal{O}(0)$, algebras over which (see

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next section) have no unit. There are also good examples of operads having only $n \geq 2$ components $\mathcal{O}(n)$.

An equivalent definition of an operad may be given in terms of operations $f \circ_i g = \gamma(f; \mathrm{id}, \dots, \mathrm{id}, g, \mathrm{id}, \dots, \mathrm{id})$, $i = 1, \dots, m$, for $f \in \mathcal{O}(m), g \in \mathcal{O}(n)$. Then the associativity condition translates as $f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i+j-1} h$ plus a natural symmetry condition for $(f \circ_i g) \circ_j h$, when g and h "fall into separate slots" in f, see *e.g.*, [**?**].

Example 3 (The Riemann surface and the endomorphism operads)**.** $\mathcal{P}(n)$ is the space of Riemann spheres with $n + 1$ boundary components, *i.e.*, n inputs and 1 output. Another example is the *endomorphism operad of a vector space* V: $\mathcal{E}nd_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$, the space of n-linear mappings from V to V.

1.1. Algebras over an operad.

Definition 4. An *algebra over an operad* $\mathcal{O}$ (in other terminology, a *representation of an operad*) is a morphism of operads $\mathcal{O} \to \mathcal{E}nd_V$, that is, a collection of maps

$$\mathcal{O}(n) \to \mathcal{E}nd_V(n) \qquad \text{for } n \geq 0$$

compatible with the symmetric group action, the unit elements, and the compositions. If the operad $\mathcal{O}$ is an operad of vector spaces, then we would usually require the morphism $\mathcal{O} \to \mathcal{E}nd_V$ to be a morphism of operads of vector spaces. Otherwise, we would think of this morphism as a morphism of operads of sets. Sometimes, we may also need a morphism to be continuous or respect differentials, or have other compatibility conditions.

1.1.1. *The commutative operad.* The *commutative operad* is the operad of k-vector spaces with the nth component $\mathcal{C}omm(n) = k$ for all $n \geq 0$. We assume that the symmetric group acts trivially on k and the compositions are just the multiplication of elements in the ground field k. An algebra over the commutative operad is nothing but a commutative associative algebra with a unit, as we see from the following exercise.

Another version of the commutative operad is $\mathcal{C}omm(n) = \text{point}$ for all $n \geq 0$. This is an operad of sets. An algebra over it is also the same as a commutative associative unital algebra.

Exercise 1. Show that the operad $\mathcal{T}op(n) = \{$the set of diffeomorphism classes of Riemann spheres with n input holes and 1 output hole$\}$ is isomorphic to the commutative operad of sets.

Exercise 2. Prove that the structure of an algebra over the commutative operad $\mathcal{C}omm$ on a vector space is equivalent to the structure of a commutative associative algebra with a unit.