LECTURE 8: OPERADS VIA GENERATORS AND RELATIONS

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1. Operads via generators and relations

The tree operads that we looked at above, such as the associative and the Lie operads, are actually operads defined by generators and relations. Here is a way to define such operads in general. To fix notation, assume throughout this section that we work with operads $\mathcal{O}(n)$, $n \geq 1$, of vector spaces.

Definition 1. An *ideal* in an operad $\mathcal{O}$ is a collection $\mathcal{I}$ of S_n-invariant subspaces $\mathcal{I}(n) \subset \mathcal{O}(n)$, for each $n \geq 1$, such that whenever $i \in \mathcal{I}$, $\gamma(\dots, i, \dots) \in \mathcal{I}$.

The intersection of an arbitrary number of ideals in an operad is also an ideal, and one can define the ideal generated by a subset in $\mathcal{O}$ as the minimal ideal containing the subset.

Definition 2. For an operad ideal $\mathcal{I} \subset \mathcal{O}$, the *quotient operad* $\mathcal{O}/\mathcal{I}$ is the collection $\mathcal{O}(n)/\mathcal{I}(n)$, $n \geq 1$, with the structure of operad induced by that on $\mathcal{O}$.

The *free operad* $F(S)$ *generated by a collection* $S = \{S(n) \mid n \geq 1\}$ *of sets*, is defined as follows.

$$F(S)(n) = \bigoplus_{n\text{-trees } T} k \cdot S(T),$$

where the summation runs over all planar rooted trees T with n labeled leaves and

$$S(T) = \operatorname{Map}(v(T), S),$$

the set of maps from the set $v(T)$ of vertices of the tree T to the collection S assigning to a vertex v with $\operatorname{In}(v)$ incoming edges an element of $S(\operatorname{In}(v))$ (the edges are directed toward the root). In other words, an element of $F(S)(n)$ is a linear combination of planar n-trees whose vertices are decorated with elements of S. There is a special tree with no vertices:

The component $F(S)(1)$ contains, apart from $S(1)$, the one-dimensional subspace spanned by this tree.

The following data defines an operad structure on $F(S)$.

1. (1) The identity element is the special tree in $F(S)(1)$ with no vertices.
2. (2) The symmetric group S_n acts on $F(S)(n)$ by relabeling the inputs.
3. (3) The operad composition is given by grafting the roots of trees to the leaves of another tree. No new vertices are created.

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Definition 3. Now let R be a subset of $F(S)$, *i.e.*, a collection of subsets $R(n) \subset F(S)(n)$. Let (R) be the ideal in $F(S)$ generated by R. The quotient operad $F(S)/(R)$ is called the *operad with generators S and defining relations R.*

Example 4. The associative operad $\mathcal{A}ssoc$ is the operad generated by a point $S = S(2) = \{\bullet\}$ with a defining relation given by the associativity condition, see Section **??**, expressed in terms of trees.

Example 5. The Lie operad $\mathcal{L}ie$ is the operad also generated by a point $S = S(2) = \{\bullet\}$ with defining relations given by the skew symmetry and the Jacobi identity, see Section **??**.

Example 6. The Poisson operad is the operad also generated by a two-point set $S = S(2) = \{\bullet, \circ\}$ with defining relations given by the commutativity and the associativity for simple trees decorated only with $\bullet$'s, the skew symmetry and the Jacobi identity for simple trees decorated with $\circ$'s, and the Leibniz identity for binary 3-trees with mixed decorations, see Section **??**.