LECTURE 9: THE A_∞ OPERAD AND A_∞-ALGEBRAS

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1. Homotopy algebra

The idea of a *homotopy "something" algebra* is to relax the axioms of the *"something" algebra*, so that the usual identities are satisfied up to homotopy. For example in a homotopy associative algebra, the associativity identity looks like

$$(ab)c - a(bc) \text{ is homotopic to zero.}$$

Or in a homotopy Gerstenhaber (G-) algebra, the Leibniz rule is

$$[a, bc] - [a, b]c \mp b[a, c] \text{ is homotopic to zero.}$$

Usually, a homotopy something algebra arises when one wants to lift the structure of a something algebra *a priori* defined on cohomology to the level of cochains.

This kind of relaxation seems to be too lax for many, practical and categorical, purposes, and one usually requires that the null-homotopies, regarded as new operations, satisfy their own identities, up to their own homotopy. These homotopies should also satisfy certain identities up to homotopy and so on. This resembles Hilbert's chains of syzygies in early homological algebra. Algebras with such chains of homotopies are called *strongly homotopy "something" algebras* or *"something"$_\infty$-algebras.*

Operads are especially helpful when one needs to work with something$_\infty$-algebras. We already know that defining the class of something algebras is equivalent to defining the something operad. Thus, if we have an operad $\mathcal{O}$, what is $\mathcal{O}_\infty$, *the* corresponding strongly homotopy operad? Markl's paper [**?**] provides a satisfactory answer to this question: the *operad* $\mathcal{O}_\infty$ is a mininal model of the operad $\mathcal{O}$. A minimal model is unique up to isomorphism. The idea is borrowed from Sullivan's rational homotopy theory; a minimal model is, first of all, a free resolution of $\mathcal{O}$ in the category of operads of complexes, *i.e.*, an operad of complexes free as an operad of graded vector spaces, whose cohomology is $\mathcal{O}[0]$, the operad $\mathcal{O}$ sitting in degree zero, if it is an operad of vector spaces, or the operad $\mathcal{O}$ sitting in the original degrees, if it is already an operad of graded vector spaces. Second of all, a minimal model must satisfy a minimality condition: its differential must be decomposable.

For certain specific classes of operads, one manages to describe a minimal model explicitly. For example, Ginzburg and Kapranov [**?**] do it (even earlier than the notion of a minimal model for an operad surfaced) for the so-called Koszul operads. Below we describe an example of such kind, giving rise to the notion of an A_∞-algebra and the A_∞ operad.

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1.1. A_∞-algebras.

Definition 1. An A_∞-*algebra*, or a *strongly homotopy associative algebra*, is a complex $V = \sum_{i\in\mathbb{Z}} V^i$ with a differential d, $d^2 = 0$, of degree 1 and a collection of n-ary operations, called *products*:

$$M_n(v_1, \dots, v_n) \in V, \qquad v_1, \dots, v_n \in V,\ n \geq 2,$$

which are homogeneous of degree $2 - n$ and satisfy the relations

$$(1.1)\quad dM_n(v_1, \dots, v_n) - (-1)^n \sum_{i=1}^{n} \epsilon(i) M_n(v_1, \dots, dv_i, \dots, v_n)$$
$$= \sum_{\substack{k+l=n+1\\ k,l\geq 2}} \sum_{i=0}^{k-1} (-1)^{i+l(n-i-l)} \sigma(i) M_k(v_1, \dots, v_i, M_l(v_{i+1}, \dots, v_{i+l}), \dots, v_n),$$

where $\epsilon(i) = (-1)^{|v_1|+\dots+|v_{i-1}|}$ is the sign picked up by taking d through $v_1, \dots, v_{i-1}$, $|v|$ denoting the degree of $v \in V$, and $\sigma(i)$ is the sign picked up by taking M_l through $v_1, \dots, v_i$.

It is remarkable to look at these relations for $n = 2$ and 3:

$$dM_2(v_1, v_2) - M_2(dv_1, v_2) - (-1)^{|v_1|} M_2(v_1, dv_2) = 0,$$

$$dM_3(v_1, v_2, v_3) + M_3(dv_1, v_2, v_3) + (-1)^{|v_1|} M_3(v_1, dv_2, v_3)$$
$$+ (-1)^{|v_1|+|v_2|} M_3(v_1, v_2, dv_3)$$
$$= M_2(M_2(v_1, v_2), v_3) - M_2(v_1, M_2(v_2, v_3)),$$

which mean that the differential d is a derivation of the bilinear product M_2 and the trilinear product M_3 is a homotopy for the associativity of M_2, respectively.

A_∞-algebras can be described as algebras over a certain tree operad. This operad is the tree part of the graph complex, which will be the topic of the following sections.

1.2. **The A_∞ operad.** Let $A_\infty(n)$ be the linear span of the set of equivalence classes of connected planar trees that have a root edge and n leaves labeled by integers 1 through n, with vertices of a valence at least 3, $n \geq 2$. For $n = 1$ take one tree with a unique edge connecting a leaf and a root. Let us not include anything for $n = 0$, although one could do that similar to the associative operad case, so that the corresponding notion of an A_∞-algebra would have a unit.

We grade each vector space $A_\infty(n)$ by defining the degree $|T|$ of a tree $T \in A_\infty(n)$ via

$$|T| := v(T) + 1 - n = e(T) + 1 - 2n,$$

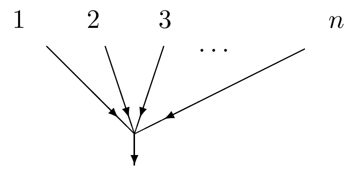
where $v(T)$ is the number of vertices and $e(T)$ the number of edges of T. Notice that $2 - n \leq |T| \leq 0$ for $n \geq 1$.

Let us define an operad structure on these spaces of trees. The symmetric group acts by relabeling the leaves, and the operad composition is obtained by *grafting*, as in the examples above, except one needs to take a sign into account. When we graft a tree T_2 to the ith leaf of a tree T_1, the result must be the grafted tree multiplied by a sign, which is (-1) to the power $(e(T_2) - 1)$(the number of edges to the right of the ith leaf in T_1), where the edges to the right of a leaf are the edges which are strictly on the right-hand side of a unique path from the leaf to the root.

The reason for the sign above is that grafting must respect the differential, which is introduced below.

Exercise 1. Show that this operad is a free operad of vector spaces generated by the following trees for $n \geq 2$, which are sometimes called *corollas*.

$\delta_n :=$



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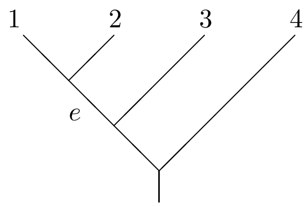
Remark 2. There is no need to mark directions on the edges of a tree: from now on we will assume the edges are directed from top to bottom.

1.3. **The tree complex.** The above operad of trees is not yet the A_∞-operad, but only its underlying operad of graded vector spaces. The $A - \infty$-operad is a DG operad, *i.e.*, an operad of complexes. The DG structure, or a differential, is defined as follows.

Before defining it, we will define the operation of internal-edge contraction on the set of trees.

Definition 3. We use the notation T/e to denote the tree obtained from a tree T by contracting an internal edge e:



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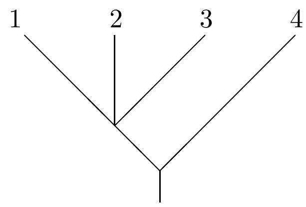


T/e

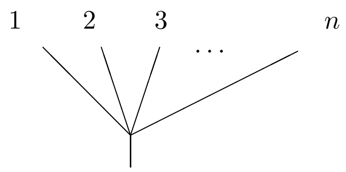
We can now define a *differential* $d : A_\infty(n) \to A_\infty(n)$ by the formula

$$dT := \sum_{T' : T = T'/e} \epsilon T',$$

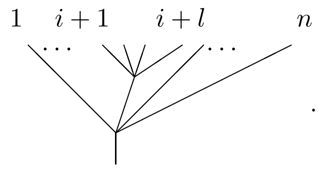
where ϵ is the sign given by counting the number of edges below or to the left of the edge e in the tree T', not counting the root.

In particular,

(1.2) d



$$= \sum_{\substack{k+l=n+1\\k,l\geq 2}} \sum_{i=0}^{k-1} (-1)^i$$



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Proposition 4. (1) *The operator d satisfies $d^2 = 0$ and $\deg d = 1$.*

(2) *The operad structure on $A_\infty = \{A_\infty(n) \mid n \geq 1\}$ is compatible with the differential d:*

$$d(T_1 \circ_i T_2) = dT_1 \circ_i T_2 + (-1)^{|T_1|} T_1 \circ_i dT_2,$$

i.e., *A_∞ is a DG operad.*

Definition 5. We will call the DG operad A_∞ the A_∞ *operad.*

Remark 6. The complex $A_\infty(n)$ is part of Kontsevich's graph cochain complex. A similar operad L_∞, based on abstract, *i.e.*, nonplanar trees, was introduced by Hinich and Schechtman [**?**]. The operad A_∞ is the dual cobar operad in the sense of Ginzburg and Kapranov [**?**] of the associative operad $\mathcal{A}ssoc$. They also show that the cohomology of the operad A_∞ is the associative operad $\mathcal{A}ssoc$ of Section **??**, implying that A_∞ is a free, and in fact, minimal, resolution of $\mathcal{A}ssoc$.

The following theorem shows that the A_∞ operad describes the class of A_∞-algebras.

Theorem 7 ([**?**])**.** *An algebra over the A_∞ operad is an A_∞-algebra. Each A_∞-algebra admits a natural structure of an algebra over the A_∞ operad.*

Proof. For a complex V of vector spaces with a differential d of degree 1, $d^2 = 0$, the structure of an algebra over the operad A_∞ on V is a morphism of DG operads:

$$\phi : A_\infty(n) \to \mathcal{E}nd_V(n), \qquad n \geq 1,$$

where $\mathcal{E}nd_V(n) := \mathrm{Hom}(V^{\otimes n}, V)$ is the *endomorphism operad*, which is also a DG operad (with the usual internal differential determined by d). Given such a morphism ϕ, we define the n-ary product on V:

$$M_n(v_1, \dots, v_n) := \phi(\delta_n)(v_1 \otimes \dots \otimes v_n).$$

Note that the degree of the product is equal to that of the corolla δ_n, which is $2 - n$. Since ϕ is a morphism of DG operads, $d\phi = \phi d$, and in view of (1.2), this is equivalent to the identity (1.1).

Conversely, given a collection of n-ary brackets on V, $n \geq 2$, we define a morphism ϕ on the generators δ_n by the above formula. The A_∞ operad is freely generated by the corollas δ_n, with a differential defined by (1.2), so the mappings ϕ define a morphism of DG operads, if the relations (1.2) are satisfied by the $\phi(\delta_n)$'s. Equations (1.1) show that this is the case. ∎