MTH G375: TOPICS IN TOPOLOGY: OPERADS
PROBLEM SET 1, DUE FEBRUARY 9, 2004

SASHA VORONOV

I encourage you to cooperate with each other on the homeworks.

Problem 1. Check that for a quasi-triangular bialgebra A (see Lecture 1 of my Minnesota lecture notes) with $R_{12}R_{21} = 1$, the category A-mod of A-modules is a symmetric monoidal category.

Problem 2. Deduce the axioms of a PROP (in terms of the symmetric group action, horizontal and vertical compositions, and the unit) from those of a tensor category with objects $\mathbb{Z}_+$ and the tensor law given by $m \otimes n = m + n$ on the objects and the associativity equivalence equal to identity:

$$\alpha = \mathrm{id}_{m+n+k} : m \otimes (n \otimes k) \to (m \otimes n) \otimes k.$$

Problem 3. A CFT with arbitrary central charge relaxes the sewing axiom to the following one: the operator $|\Sigma_2 \circ \Sigma_1\rangle$ corresponding to the result of sewing two Riemann surfaces Σ_1 and Σ_2 is equal to the composition $|\Sigma_2\rangle \circ |\Sigma_1\rangle$ up to a nonzero scalar factor $\lambda(\Sigma_1, \Sigma_2)$:

$$|\Sigma_2 \circ \Sigma_1\rangle = \lambda(\Sigma_1, \Sigma_2)|\Sigma_2\rangle \circ |\Sigma_1\rangle.$$

This scalar λ generalizes the notion of a two-cocycle of $\mathrm{Diff}(S^1)$. Find out an equation of this type on λ. [See any textbook on group cohomology, e.g., Group Cohomology by Kenneth Brown, to find a two-cocycle equation.] Is this the only condition λ must satisfy?

Problem 4. Prove the following algebraic lemma, used in identifying a TFT as a Frobenius algebra. Let V be a vector space over a field k, $\alpha : k \to V \otimes V$ and $\beta : V \otimes V \to k$ two linear maps, which are symmetric as tensors, i.e., $\alpha(1)$ is a symmetric element of $V \otimes V$ and β is a symmetric bilinear form on V. Suppose that the following composite map is equal to the identity:

$$V \xrightarrow{\mathrm{id} \otimes \alpha} V \otimes V \otimes V \xrightarrow{\beta \otimes \mathrm{id}} V.$$

Prove that V is finite-dimensional and β is nondegenerate.

Problem 5. Prove carefully that if A is a Frobenius algebra, then one can define a TFT with A as a state space with the multiplication corresponding to the pair of pants and the trace to the cap.

Problem 6 (Dijkgraaf-Witten's toy model)**.** Define a TFT, starting from a finite group G. Take the state space A to be the algebra of complex-valued functions on the finite set of isomorphism classes of principal G-bundles over the circle S^1. Identify this set with $\mathrm{Hom}(\pi_1(S^1), G)/\mathrm{Ad}G = \mathrm{Hom}(\mathbb{Z}, G)/\mathrm{Ad}G = G/\mathrm{Ad}G$, where $\mathrm{Ad}G$ is the adjoint action of G on itself, i.e., the action by conjugation. Imply that

Date: January 27, 2004.

A may be identified as an algebra with the center of the group algebra $\mathbb{C}[G]$. If Σ is a cap (hemisphere) with 1 input, let

$$|\Sigma\rangle(f) := \sum_P \frac{1}{|\mathrm{Aut} P|} f(P|_\partial),$$

where the summation runs over the set of the isomorphism classes of principal G-bundles over Σ and f is a complex-valued function on the set of isomorphism classes of principal G-bundles over the input $\partial\Sigma \cong S^1$. Show that the resulting trace $\theta : A \to \mathbb{C}$ is given by $\theta(\sum_{g\in G} c_g g) = c_e/|G|$. [Hint: If you feel uneasy with principal G-bundles, consult a math encyclopedia, your favorite book on geometry/topology, or just take $\mathrm{Hom}(\pi_1(\Sigma), G)/\mathrm{Ad}G$ as the definition of the set of isomorphism classes of principle G-bundles over Σ.]