MTH G375: TOPICS IN TOPOLOGY: OPERADS
PROBLEM SET 2, DUE MARCH 8, 2004

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I encourage you to cooperate with each other on the homeworks.

Problem 1. Prove that an algebra over the PROP whose spaces $P(m, n)$, $m, n \geq 1$, consist of just one point, in the category of sets is a one-point set.

Problem 2. Prove that an algebra over the operad whose spaces $O(n)$, $n \geq 1$, consist of just one point, in the category of sets is an abelian monoid (without a unit). Deduce that any abelian monoid is an algebra over any operad in the category of sets.

Problem 3. Show that an A_∞-algebra structure on a graded vector space V is equivalent to the structure of a differential on the tensor coalgebra $T^c(V[1])$. Here $V[1]$ is the "desuspension" of the graded vector space $V = \bigoplus_i V^i$: $V[1]^i := V^{i+1}$, $T^c(V[1]) := \bigoplus_{n\geq 0} V[1]^{\otimes n}$ as a graded vector spaces, with a comultiplication $\Delta(x_1 \otimes \cdots \otimes x_n) := \sum_{i=0}^n (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n)$. A *differential* on a graded coalgebra is an operator D of degree one with properties $D^2 = 0$ (a differential on a graded vector space $T^c(V[1])$), $D(1) = 0$, $\Delta(Dx) = D\Delta(x) := x_{(1)} \otimes x_{(2)} + (-1)^{|x_{(1)}|} x_{(1)} \otimes Dx_{(2)}$ (a derivation of the graded coalgebra structure). [Hint: First, show that a derivation $D : T^c(V[1]) \to T^c(V[1])$ of the tensor coalgebra is determined by its projection from $T^c(V[1])$ onto $V[1]$. Then brake D into a sum $\sum_{n\geq 0} m_n$, where $m_n : (V[1])^{\otimes n} \to V[1]$.]

Problem 4. Prove that the top-degree cohomology of the A_∞ operad is the "associative" operad of Lecture 7 of my Minnesota lecture notes.

Problem 5. Describe algebraically an algebra over the operad $\mathcal{L}ie$ modified to include a 0-tree (no leaves, no vertices, but one root), whose composition with any other tree is defined as (a) zero, (b) killing the leaf to which the 0-tree is grafted and erasing the adjacent vertex.

Problem 6. Show that $\mathcal{C}om^! = \mathcal{L}ie$. [Hint: this is a simple computation of linear algebra.]

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