UNIVERSITY OF MINNESOTA

This is to certify that I have examined this bound copy of a doctoral thesis by

Philip Lloyd Osterlund

and have found it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Name of Faculty Adviser

Signature of Faculty Adviser

Date

GRADUATE SCHOOL

Tensor Decompositions of the Regular Representation of p-Groups Over Fields of Characteristic p

A Thesis
Submitted to the Faculty of The Graduate School
of The University of Minnesota
By

Philip Lloyd Osterlund

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

December, 1997

Chapter 0: Introduction

In this thesis we investigate certain properties of group rings. We describe a procedure for constructing and classifying all uniserial modules for the group ring of a p-group over a field of characteristic p. We also discuss non-trivial decompositions of the regular representation of a group ring into a tensor product of modules. We begin by explaining the context in which we will work, and we describe the less familiar notation.

Throughout this work, p is a prime and $\mathbb{F}$ is a field of characteristic p. Unless stated otherwise, G is a p-group, and we study the group ring $\mathbb{F}G$ and its modules. We will almost always consider left modules. The group ring $\mathbb{F}G$ acts on itself as a module by left multiplication. This module is the *regular representation* of $\mathbb{F}G$, which we also denote $\mathbb{F}G$.

For a subgroup H of G, we will often wish to refer to the sum of the elements of H in $\mathbb{F}G$, and we denote this by $\|H\|$. The sum of the elements of a coset gH of H in G is denoted as $g\|H\|$ or $\|gH\|$.

A group ring is a particular type of algebra. Let A be a finite dimensional algebra. A *simple* A-module S is a non-zero module which contains no submodules other than 0 and itself. A *semisimple* A-module is a module which is a direct sum of simple modules.

The *radical* of A, written $\operatorname{Rad}(A)$, is the intersection of all the maximal left (or right) ideals of A. If M is an A-module then we define $\operatorname{Rad}(M) = \operatorname{Rad}(A)M$. Powers of the radical are defined as $\operatorname{Rad}^0(M) = M$, and $\operatorname{Rad}^n(M) = \operatorname{Rad}(A)\operatorname{Rad}^{n-1}(M) = \operatorname{Rad}^n(A)M$. The socle of M, written $\operatorname{Soc}(M)$, is the set of elements annihilated by $\operatorname{Rad}(A)$, i.e. $\operatorname{Soc}(M) = \{m \in M \bigm| \operatorname{Rad}(A)m = 0\}$. Powers of the socle are defined as $\operatorname{Soc}^n(M) = \{m \in M \bigm| \operatorname{Rad}^n(A)m = 0\}$.

The *radical series* of the module M is the series of modules,

$$M \supseteq \operatorname{Rad}(M) \supseteq \operatorname{Rad}^2(M) \supseteq \cdots.$$

This is the fastest descending series of submodules with semisimple quotients. The *socle series* is described similarly, and is the fastest ascending series of submodules with semisimple quotients.

The *Loewy length* of a module M, written $\ell\ell(M)$, is the minimal value r such that $\mathrm{Rad}^r(M) = 0$, which is also equal to the minimal value r such that $\mathrm{Soc}^r(M) = M$. Both the radical series and the socle series of M have length $\ell\ell(M)$. The maximum possible Loewy length of an A-module is $L = \ell\ell(A)$.

A *uniserial* module M is a module for which the following equivalent statements hold:

1) M has a unique composition series;
2) The submodules of M are totally ordered under inclusion;
3) The successive quotients of the radical series are simple or zero;
4) The successive quotients of the socle series are simple or zero.

We study uniserial modules for several reasons. They have appeared widely in the literature on the representation theory of finite-dimensional algebras over many years. See for example [F, B, ZH, AR&S]. In the context of group representation they have been studied in [Sh, LG], where they play an essential role in describing p-group structure. Also they were studied in [P].

In this thesis we show in Lemma 1.4 of Chapter 1 that each uniserial module is isomorphic to a cyclic submodule of the regular representation. This fact makes it possible to classify all uniserial modules up to isomorphism for any given finite dimensional group ring. Our work here can be viewed as treating a special case of open problems (1)-(4) on page 411 of [AR&S], and we present explicit classifications of uniserial modules for many groups of order 16. We are able to apply these results using the fact that uniserial modules are prime candidates for factors in a tensor decomposition of the regular representation of a group. We describe such decompositions explicitly in many cases.

In Chapter 1 we focus on classifying uniserial modules. In Section 1 we cover much of the theory behind the classification of uniserial modules. Some of the theory applies more broadly to cyclic modules. In Section 2 we outline a procedure for finding all uniserial submodules of a group ring. Section 3 consists of an example of the use of this procedure. We determine the set of all uniserial submodules of $\mathbb{F}Q_8$, and classify all uniserial modules of $\mathbb{F}Q_8$ up to isomorphism.

Chapter 2 consists of calculation results, using the algorithm described in Chapter 1. Each section concentrates on a specific non-abelian group G of order 16. We restrict our attention to the groups that may not be written as the direct product of non-trivial subgroups of G. For each of these groups, we find the set of all uniserial submodules of $\mathbb{F}G$, and classify all uniserial modules of $\mathbb{F}G$, where $\mathbb{F}$ is a field of characteristic 2.

In Chapter 3 we address the issue of tensor decompositions of the regular representation. We may take our motivation for the study of such decompositions in part from the simple observation for a direct product of groups that $\mathbb{F}[G \times H] \cong \mathbb{F}G \otimes \mathbb{F}H$. One may ask more generally when such a decomposition occurs, and this question is resolved by Carlson and Kovács [C&K] when G is an abelian p-group and the decomposition is a tensor product of rings. Such decompositions also have relevance for the construction of multiple complexes, as considered by Benson and Carlson [B&C].

In Section 1 we develop some of the theory of tensor decompositions. In Section 2 we concentrate on permutation modules. This is another important class of modules which present themselves as factors in tensor decompositions. In Section 3, we use our results of Chapter 2 and the theory developed earlier in this chapter to determine all tensor decompositions of $\mathbb{F}G$, where G is a group considered in Chapter 2.

We will make extensive use of the theory developed by Jennings [J] which we now summarize. Some of the notation is found in [Al 1].

For a p-group G, Jennings [J, Al 1, Sc] describes a decreasing series of subgroups, $\kappa_\lambda(G) = \{g \in G \bigm| g \equiv 1 \text{ modulo } \mathrm{Rad}^\lambda(\mathbb{F}G)\}$, which is sometimes called the Jennings series. This series of subgroups has the following properties:

1) $[\kappa_\lambda, \kappa_\mu] \subseteq \kappa_{\lambda+\mu}$,
2) $g^p \in \kappa_{ip}$ for all $g \in \kappa_i$,
3) $\kappa_\lambda / \kappa_{2\lambda}$ is elementary abelian.

Furthermore, we may generate κ_λ as

$$\kappa_\lambda = \langle [\kappa_{\lambda-1}, G], \kappa^{(p)}_{\lceil \lambda/p \rceil} \rangle,$$

where $\kappa_1 = G$, and $\lceil \lambda/p \rceil$ is the least integer greater than or equal to λ/p, and $\kappa^{(p)}_\lambda$ is the set of pth powers of elements of κ_λ. With this formulation, we have $\kappa_1(G) = G$, and it is clear that $\kappa_2(G)$ is the Frattini subgroup of G.

For each $i \geq 1$ let d_i be the dimension of κ_i/κ_{i+1}. Choose any elements $x_{i,s}$ of G such that the set $\{x_{i,s}\kappa_{i+1} \mid 1 \leq s \leq d_i\}$ forms a basis for κ_i/κ_{i+1}. Let

$\bar{x}_{i,s} = x_{i,s} - 1 \in \mathbb{F}G$. There are $|G|$ products of the form $\prod \bar{x}_{i,s}^{\alpha_{i,s}}$, where the factors are listed in lexicographic order, and $0 \leq \alpha_{i,s} \leq p-1$. The weight of such a product is defined to be $\sum i\alpha_{i,s}$. Jennings' theorem states that the set of products of weight w lie in $\mathrm{Rad}^w(\mathbb{F}G)$, and forms a basis modulo $\mathrm{Rad}^{w+1}(G)$. Also, the set of products of weight $\geq w$ forms a basis for $\mathrm{Rad}^w(\mathbb{F}G)$.

Alperin comments [Al 1] that the order of the factors is irrelevant, as long as some order is chosen. After choosing a particular order for these factors, let $\{\beta_{i,t}\}$ be the set of such products with weight i. Writing $L = \ell\ell(\mathbb{F}G)$, the element $\beta_{L-1,1} = \prod \bar{x}_{i,s}^{p-1} = \|G\|$ is a generator of the socle of $\mathbb{F}G$, and has weight $L-1$.

Often it is possible to choose an element $x_{i,s}$ such that $x_{i,s}^p$ is also one of our chosen coset representatives for κ_{ip+1} in κ_{ip}, and when this can be done we make this choice. As we do not need to choose the lexicographic ordering in the products, we choose an ordering that places $x_{i,s}$ and $x_{i,s}^p$ together.

For example, in the group $D_8 = \langle x, y | x^4 = y^2 = 1\rangle$ we have $\kappa_1 = D_8$, $\kappa_2 = \langle x^2 \rangle$, and $\kappa_3 = 1$. We choose $x_{1,1} = x$, $x_{1,2} = y$, and $x_{2,1} = x^2$. We choose to write the factors in the order, $\bar{x}^{\alpha_{1,1}}\bar{x}^{2\alpha_{2,1}}\bar{y}^{\alpha_{1,2}} = \bar{x}^{\alpha_{1,1}+2\alpha_{2,1}}\bar{y}^{\alpha_{1,2}}$, where $0 \leq \alpha_{i,s} < 2$. The elements are then of the form $\bar{x}^i\bar{y}^j$ with $0 \leq i < 4$, and $0 \leq j < 2$.

Chapter 1: Classifying Uniserial Modules

In this chapter, after a few preliminary lemmas, we describe a procedure for classifying all uniserial modules of the group ring for a p-group. We then present an example to illustrate the use of this procedure.

§1 Uniserial Modules

The first four lemmas are found in the literature. See, for example, [Al 2], [H], and [AR&S].

We use the *augmentation map* of a group ring, which is the map, $\epsilon : \mathbb{F}G \to \mathbb{F}$ given by

$$\epsilon\Big(\sum_{g\in G} g a_g\Big) = \sum_{g\in G} a_g,$$

where a_g is an element of the field $\mathbb{F}$ for each element g of G. The *augmentation ideal* IG of $\mathbb{F}G$ is the kernel of this map. This ideal has codimension 1, and has the set $\{g-1 \mid g \in G, g \neq 1\}$ as a basis. This set together with the element 1 forms a basis for $\mathbb{F}G$.

Lemma 1.1. *If the group G is generated by $\{x_1, \ldots, x_n\}$, then the augmentation ideal is the two-sided ideal IG of the ring $\mathbb{F}G$ generated by the elements $\{\bar{x}_1, \ldots, \bar{x}_n\}$. Thus we may write, $IG = \mathbb{F}G\bar{x}_1 + \cdots + \mathbb{F}G\bar{x}_n$.*

Lemma 1.2. *If G is a finite p-group, and $\mathbb{F}$ is a field of characteristic p, then*

i) the trivial module is the only simple module of $\mathbb{F}G$;
ii) the radical and socle series of $\mathbb{F}G$ coincide;
iii) the radical of $\mathbb{F}G$ is the augmentation ideal, IG;
iv) the unique indecomposable projective $\mathbb{F}G$-module is $\mathbb{F}G$;
v) the unique indecomposable injective $\mathbb{F}G$-module is $\mathbb{F}G$.

Lemma 1.3. *(Nakayama) If J is an ideal in a ring R with identity, then the following are equivalent.*

i) J is contained in the Jacobson radical of R;

ii) $1_R - j$ *is a unit for every* $j \in J$

iii) If A *is a finitely generated* R*-module such that* $JA = A$*, then* $A = 0$*;*

iv) If B *is a submodule of a finitely generated* R*-module* A *such that* $A = JA+B$*, then* $A = B$*.*

We relate this lemma to our situation where G is a p-group by noting that the Jacobson radical of the ring $\mathbb{F}G$ is the augmentation ideal, IG.

Lemma 1.4. *A uniserial module* M *is a module for which the following equivalent statements hold:*

i) M *has a unique composition series;*

ii) The submodules of M *are totally ordered under inclusion;*

iii) The successive quotients of the radical series are simple or zero;

iv) The successive quotients of the socle series are simple or zero.

Lemma 1.5. *If* M *is a uniserial module for* $\mathbb{F}G$*, and* G *is a finite group, then*

i) M *is cyclic (i.e. it can be generated by a single element);*

ii) M *is a homomorphic image of* $\mathbb{F}G$*;*

iii) M *is isomorphic to a submodule of* $\mathbb{F}G$*.*

Proof. Uniserial modules have the unique composition series

$$M = \mathrm{Rad}^0(M) \supset \mathrm{Rad}(M) \supset \ldots \supset \mathrm{Rad}^L(M) = 0.$$

An element of M must therefore generate a power of the radical as a left module. If this element is not an element of the radical, then it must generate the module M.

Any module of an algebra A that is generated by a single element is a homomorphic image of A.

The module M has a simple socle, $S = \mathrm{Soc}(M)$. The module S is isomorphic to a submodule of $\mathbb{F}G$. As $\mathbb{F}G$ is injective, the inclusion map of S into $\mathbb{F}G$ may be extended to a map ϕ on the module M. If the kernel of ϕ were non-zero, it would intersect the socle of M non-trivially. But ϕ is an extension of an injection of S, so the kernel must be 0. Thus ϕ is one-to-one, and M is isomorphic to its image in $\mathbb{F}G$. $\square$

As a consequence, we will often assume that a uniserial module is a submodule of $\mathbb{F}G$, and we designate this uniserial module as the module $\mathbb{F}GA$, where A is an element of $\mathbb{F}G$ that generates the module. The zero module is considered uniserial with Loewy length 0.

Lemma 1.6. *A non-zero module M is uniserial if and only if* $\operatorname{Rad}(M)$ *is uniserial and* $M/\operatorname{Rad}(M)$ *is simple.*

Proof. Let $R = \operatorname{Rad}(M)$. If M is a uniserial module, then M/R is simple. As R is a submodule of M, it too must have a unique composition series, and must also be uniserial. Conversely, if R is uniserial, and M/R is simple, then the successive quotients of the radical series of M are simple, so M is uniserial. □

Corollary 1.7. *Let $\mathbb{F}$ be a field of characteristic p and let G be a finite p-group generated by the elements $\{x_1, x_2, \ldots, x_n\}$. A non-zero $\mathbb{F}G$-module M is uniserial if and only if M has a uniserial submodule N with* $\dim(N) = \dim(M) - 1$ *and* $\mathbb{F}G\bar{x}_i M = N$ *for some i.*

Proof. In this case, all simple modules of $\mathbb{F}G$ are isomorphic to $\mathbb{F}$. If M is a non-zero $\mathbb{F}G$-module, then

$$\operatorname{Rad}(M) = \operatorname{Rad}(\mathbb{F}G)M = IGM = (\mathbb{F}G\bar{x}_1 + \cdots + \mathbb{F}G\bar{x}_n)M.$$

If M is uniserial, then $N = \operatorname{Rad}(M)$ must also be uniserial, and M/N must be simple. The modules $\mathbb{F}G\bar{x}_j M$ are all submodules of $\operatorname{Rad}(M)$, and generate $\operatorname{Rad}(M)$. Since the submodules of M are totally ordered under inclusion, there must be an index i such that $\mathbb{F}G\bar{x}_i M$ contains the modules $\mathbb{F}G\bar{x}_j M$ for all j. As $\operatorname{Rad}(M) = IGM$, we must have $N = \operatorname{Rad}(M) = \mathbb{F}G\bar{x}_i M$. Since $\dim(M/N) = \dim(\mathbb{F}) = 1$, we have $\dim(N) = \dim(M) - 1$.

Conversely, if $N = \mathbb{F}G\bar{x}_i M$ is a uniserial module, and $\dim(N) = \dim(M) - 1$, then clearly $N \subseteq \operatorname{Rad}(M)$, as $\bar{x} \in \operatorname{Rad}(\mathbb{F}G)$. We have $\dim(M/N) = 1$, so the quotient is simple. Thus, by the lemma, M is uniserial. □

In the following proposition we regard $\mathrm{Hom}_{\mathbb{F}G}(\mathbb{F}GA, \mathbb{F}G)$ as a right $\mathbb{F}G$-module by means of the action $(\phi \cdot C)(z) = \phi(z) \cdot C$, where $C \in \mathbb{F}G$, and $z \in \mathbb{F}G$.

Proposition 1.8. *If A is an element of $\mathbb{F}G$ then* $\mathrm{Hom}_{\mathbb{F}G}(\mathbb{F}GA, \mathbb{F}G) \cong A\mathbb{F}G$ *as a right $\mathbb{F}G$-module.*

Proof. Let $\iota : \mathbb{F}GA \to \mathbb{F}G$ be the inclusion map. Given an $\mathbb{F}G$ homomorphism $\phi : \mathbb{F}GA \to \mathbb{F}G$, by the injectivity of $\mathbb{F}G$, there is an extension $\psi : \mathbb{F}G \to \mathbb{F}G$ such that $\phi = \psi \circ \iota$. Let $B = \psi(1)$. Given an element $x \in \mathbb{F}G$, we have $\psi(x) = x\psi(1) = xB$, as ψ is an $\mathbb{F}G$-homomorphism. Thus the homomorphism acts as right multiplication by the element B. Restricting this action to $\mathbb{F}GA$, we have $\phi(xA) = \psi\iota(xA) = xAB$. If ψ' is another extension of ϕ, we have $\phi(x) = xA\psi'(1)$, so $A\psi(1) = A\psi'(1) = AB$. Thus we associate the homomorphism ϕ with the element AB. We designate this association by writing $\phi = \phi_{A,B}$.

Given an element AC of $A\mathbb{F}G$, let $\phi_{A,C} : \mathbb{F}GA \to \mathbb{F}G$ be the map given by $\phi_{A,C}(xA) = xAC$. This is an $\mathbb{F}G$-homomorphism on $\mathbb{F}GA$, and $\phi_{A,B} = \phi_{A,C}$ if and only if $AB = AC$. When adding homomorphisms we have

$$(\phi_{A,B} + \phi_{A,C})(xA) = xAB + xAC = xA(B + C) = \phi_{A,B+C}(x), \qquad \text{so}$$

$$\phi_{A,B} + \phi_{A,C} = \phi_{A,B+C}.$$

Also, $\phi_{A,B}(x) \cdot C = xABC$, so $\phi_{A,B} \cdot C = \phi_{A,BC}$.

We have shown that there is a one-to-one correspondence between elements AB of $A\mathbb{F}G$ and homomorphisms $\phi_{A,B} : \mathbb{F}GA \to \mathbb{F}G$, and that both sets act as right $\mathbb{F}G$-modules in the same manner. $\square$

Corollary 1.9. *Let G be a p-group and let A and B be elements of $\mathbb{F}G$. Then $\mathbb{F}GA \cong \mathbb{F}GB$ if and only if $\mathbb{F}GB = \mathbb{F}GAx$, where $x \in \mathbb{F}G$ is some unit.*

Proof. Assume that $\mathbb{F}GA \cong \mathbb{F}GB$. The module $\mathbb{F}GB$ is a submodule of $\mathbb{F}G$, so by Proposition 1.8 we know that there is an isomorphism $\phi_{A,x} : \mathbb{F}GA \to \mathbb{F}GB$ whose image is $\mathbb{F}GAx = \mathbb{F}GB$. The element x is not an element of $\mathrm{Rad}(\mathbb{F}G)$ because the Loewy lengths of $\mathbb{F}GA$ and $\mathbb{F}GB$ must be the same and the Loewy length of $\mathbb{F}GAx$ is determined by the power of $\mathrm{Rad}(\mathbb{F}G)$ in which Ax lies. All elements of $\mathbb{F}G$ which are not elements of the radical are units, so x is a unit.

Conversely, assume that x is a unit of $\mathbb{F}G$, and that $\mathbb{F}GB = \mathbb{F}GAx$. Then by Proposition 1.8, $\mathbb{F}GB$ is a homomorphic image of $\mathbb{F}GA$. If we apply the map $\phi_{B,x^{-1}}$ to the module $\mathbb{F}GB$, we see that $\mathbb{F}GA = \mathbb{F}GBx^{-1}$, so $\mathbb{F}GA$ is also a homomorphic image of $\mathbb{F}GB$, so the modules are isomorphic. $\square$

We now come to our first substantial result.

Theorem 1.10. *Let G be a p-group. If $A \in \mathbb{F}G$, and the left $\mathbb{F}G$-module $\mathbb{F}GA$ is uniserial of length l, then the right $\mathbb{F}G$-module $A\mathbb{F}G$ is also uniserial of length l.*

Proof. If $\mathbb{F}GA$ is of length 1, then $\mathbb{F}GA$ is the trivial module, so A is a multiple of $\|G\|$. The group ring element $\|G\|$ is central, so $\mathbb{F}GA = \mathbb{F}G\|G\| = \|G\|\mathbb{F}G = A\mathbb{F}G$, and is uniserial of length 1.

Assume the statement of the theorem is true for all $j < l$. Then if A is an element of $\mathbb{F}G$ which generates a left $\mathbb{F}G$-module of length $j < l$, then $\operatorname{Hom}(\mathbb{F}GA, \mathbb{F}G) \cong A\mathbb{F}G$, and by induction, these have dimension j.

Let $A^{(l)}$ be an element of $\mathbb{F}G$ such that $\mathbb{F}GA^{(l)}$ is uniserial of length l. Since $\mathbb{F}G$ is injective and contains every simple module in its socle, every indecomposable left $\mathbb{F}G$-module whose socle is simple is isomorphic to a submodule of $\mathbb{F}G$. Thus there is a submodule M of $\mathbb{F}G$ that is isomorphic to $\mathbb{F}GA^{(l)}/\operatorname{Soc}(\mathbb{F}GA^{(l)})$. Let ϕ be the composite homomorphism $\mathbb{F}GA^{(l)} \to \mathbb{F}GA^{(l)}/\operatorname{Soc}(\mathbb{F}GA^{(l)}) \to M$. Then ϕ corresponds to an element $A^{(l)}B \in A^{(l)}\mathbb{F}G$, for some element B of $\mathbb{F}G$, and ϕ has the form $\phi(gA^{(l)}) = gA^{(l)}B$. Thus $M = \mathbb{F}GA^{(l)}B$, and it is uniserial of length $l-1$. By our assumption, $A^{(l)}B\mathbb{F}G$ is a uniserial right $\mathbb{F}G$-module of length $l-1$.

Given $C \in \mathbb{F}G$, let $u = \epsilon(C) \in \mathbb{F}$, and $r = C - u \in \operatorname{Rad}(\mathbb{F}G)$. Then $\mathbb{F}GA^{(l)}r$ is a homomorphic image of $\mathbb{F}GA^{(l)}$. Since $A^{(l)}r$ is in a higher power of the radical than $A^{(l)}$, it is in a lower power of the socle of $\mathbb{F}G$ (since the radical and socle series of $\mathbb{F}G$ coincide when G is a p-group), and it must be true that $\ell\ell(\mathbb{F}GA^{(l)}r) < \ell\ell(\mathbb{F}GA^{(l)})$. This implies that $\mathbb{F}GA^{(l)}r$ has dimension less than l. This means that it is also a homomorphic image of $\mathbb{F}GA^{(l)}B$, so $A^{(l)}r \in A^{(l)}B\mathbb{F}G$. The element $A^{(l)}C = A^{(l)}u + A^{(l)}r$ lies in $\mathbb{F}A^{(l)} + A^{(l)}B\mathbb{F}G$, since u is an element of $\mathbb{F}$. Since C was an arbitrary element of $\mathbb{F}G$ he have $A^{(l)}\mathbb{F}G = \mathbb{F}A^{(l)} + A^{(l)}B\mathbb{F}G$, and it has dimension l. Since $A^{(l)} \in \mathrm{Soc}^l(\mathbb{F}G)$, it must be true that $\ell\ell(A^{(l)}\mathbb{F}G) = l$. This means that $A^{(l)}\mathbb{F}G$ is a uniserial module of length l. $\square$

We have shown that the set of homomorphisms from a uniserial left $\mathbb{F}G$-module into $\mathbb{F}G$ forms a uniserial right $\mathbb{F}G$-module of the same length.

Lemma 1.11. *Let G be a p-group and let A and B be two non-zero elements of $\mathbb{F}G$. The cyclic modules $\mathbb{F}GA$ and $\mathbb{F}GB$ are equal if and only if $B = fA + r$ for some non-zero element f of $\mathbb{F}$ and some element r of $\mathrm{Rad}(\mathbb{F}GA)$.*

Proof. Assume $\mathbb{F}GA = \mathbb{F}GB$. As $B \in \mathbb{F}GA$, we may write $B = xA$, for some element x of $\mathbb{F}G$. The radical of $\mathbb{F}G$ is a subspace of codimension 1, so we may write $x = f1 + s$, for some element f of $\mathbb{F}$, and some element s of $\mathrm{Rad}(\mathbb{F}G)$. Thus $B = (f1 + s)A = fA + r$, where $r = sA \in \mathrm{Rad}(\mathbb{F}GA)$. The element f must be non-zero because $\mathbb{F}Gr \subseteq \mathrm{Rad}(\mathbb{F}GA) \neq \mathbb{F}GA$.

Assume that $B = fA + r$, for some non-zero element $f \in \mathbb{F}$ and some element $r \in \mathrm{Rad}(\mathbb{F}GA)$. Then $A \in \mathbb{F}GB + \mathrm{Rad}(\mathbb{F}GA)$ so $\mathbb{F}G = \mathbb{F}GB + \mathrm{Rad}(\mathbb{F}GA)$. Therefore by Nakayama's lemma $\mathbb{F}GA = \mathbb{F}GB$. $\square$

Lemma 1.12. *Let G be a p-group. If A and B are elements of $\mathbb{F}G$, $\lambda = \ell\ell(\mathbb{F}GA)$, and $\mathbb{F}GA \cong \mathbb{F}GB$, then $B = fA + r$ for some non-zero element f of $\mathbb{F}$ and some element $r \in \mathrm{Soc}^{\lambda-1}(\mathbb{F}G)$.*

Proof. Assume that $\mathbb{F}GA \cong \mathbb{F}GB$. By Corollary 1.9 we know that $\mathbb{F}GB = \mathbb{F}GAx$ for some $x \in \mathbb{F}G$. We rewrite $x = f_1 + s_1$, where $0 \neq f_1 \in \mathbb{F}$ and $s_1 \in \mathrm{Rad}(\mathbb{F}G)$. Then $\mathbb{F}GB = \mathbb{F}GA(f_1 + s_1)$. Since $f_1 \neq 0$, we have $Af_1 \in \mathrm{Soc}^{\lambda}(\mathbb{F}G) - \mathrm{Soc}^{\lambda-1}(\mathbb{F}G)$. As $s_1 \in \mathrm{Rad}(\mathbb{F}G)$, and $\mathrm{Soc}^{\lambda}(\mathbb{F}G)$ is a two-sided ideal, we have $As_1 \in \mathrm{Soc}^{\lambda-1}(\mathbb{F}G)$. By the previous lemma, we must have a non-zero element $f_2 \in \mathbb{F}$ and an element $r_2 \in \mathrm{Rad}(\mathbb{F}GA(f_1+s_1))$ such that $B = f_2A(f_1+s_1)+r_2 = fA+r$, where $f = f_2f_1$ and $r = f_2As_1+r_2$. As $A \subseteq \mathrm{Soc}^{\lambda}(\mathbb{F}G)$, we have $r \in \mathrm{Soc}^{\lambda-1}(\mathbb{F}G)$. $\square$

We use these results to justify the algorithm described in the next section.

§2 A Procedure for Finding Uniserial Modules

We are about to describe an algorithm for finding uniserial modules of $\mathbb{F}G$. The algorithm works when G is a finite p-group and $\mathbb{F}$ is a field of characteristic p, and we make these assumptions. We choose a set of generators of G, $\{x_1, \ldots, x_n\}$.

Algorithm 2.1. *Suppose we are given a uniserial module $N \subseteq \mathbb{F}G$ of length $l-1$. We construct all uniserial submodules of $\mathbb{F}G$ having length l which contain N as a submodule by the following procedure:*

We find all elements $A \in \mathbb{F}G$ with the properties

i) $\bar{x}_i A \in N, \forall i$

ii) for some i, $\bar{x}_i A$ generates N.

Then the module $\mathbb{F}GA$ is uniserial of the kind specified. Every uniserial module may be constructed in this way.

Remark 2.2. Such A must necessarily lie in $\mathrm{Soc}^l(\mathbb{F}G)$. If we start with the submodule 0 of $\mathbb{F}G$, we generate all uniserial submodules of $\mathbb{F}G$ by successive use of Algorithm 2.1. Since every uniserial module is isomorphic to a submodule of $\mathbb{F}G$, this algorithm produces all isomorphism types of uniserial modules of $\mathbb{F}G$. Note, however, that distinct uniserial submodules of $\mathbb{F}G$ may in fact be isomorphic. We explain later how to distinguish between isomorphism classes of uniserial modules.

Let A be an element of the kind specified in the algorithm, and let $M = \mathbb{F}GA$. Then if $0 \neq f \in \mathbb{F}$, and $r \in \mathrm{Rad}(M) = N$, then $fA + r$ also generates M. We would like to find one *distinguished generating element* for each module. We may then classify all uniserial modules of $\mathbb{F}G$ having length l by the set of distinguished generating elements.

A description of the distinguished generating element of each module will now be given.

Let $\mathcal{B} = \{b_{i,\lambda}\}$ be a basis of $\mathbb{F}G$ such that $\mathcal{B}_\lambda = \{b_{i,\mu} | \mu \leq \lambda\}$ is a basis for $\mathrm{Soc}^\lambda(\mathbb{F}G)$. We will call such a basis a *filtered basis* of $\mathbb{F}G$. Among the basis elements with a common second subscript, set up an ordering that is reflected in the first subscript. Write elements of $\mathbb{F}G$ as linear combinations of these basis elements.

Lemma 2.3. *If M is a uniserial submodule of $\mathbb{F}G$ with Loewy length l, then there are unique sets $\{A^{(\lambda)}|1 \leq \lambda \leq l\} \subseteq M$ and $\{b_{i_\lambda,\lambda}|1 \leq \lambda \leq l\} \subseteq \mathcal{B}$ such that*

i) $\mathrm{Soc}^\lambda(M) = \mathbb{F}GA^{(\lambda)}$

ii) i_λ is the minimal i such that the coefficient of $b_{i,\lambda}$ is non-zero in $A^{(\lambda)}$

iii) The coefficient of $b_{i_\lambda,\lambda}$ in $A^{(\lambda)}$ is 1

iv) The coefficient of $b_{i_\lambda,\lambda}$ in $A^{(\mu)}$ is 0 for all $\mu > \lambda$.

Proof. If $l = 1$, then the only choice is $A^{(1)} = b_{1,1} = ||G||$. We use induction on the Loewy length of M. Assume the statement is true for $N = \mathrm{Rad}(M) \neq 0$. The unique sets $\{A^{(\lambda)} \mid 1 \leq \lambda < l\}$ and $\{b_{i_\lambda,\lambda} \mid 1 \leq \lambda < l\}$ associated with N then exist. Let A be a generator of M. As M is uniserial, M/N is isomorphic to the trivial module. The choice of $b_{i_l,l}$ must therefore be unique, regardless of the generator A for M. Let $A_1 = A/c$, where c is the coefficient of $b_{i_l,l}$ in A. Let c_λ be the coefficient of $b_{i_\lambda,\lambda}$ in A_1, where $1 \leq \lambda < l$. Let

$$A^{(l)} = A_1 - \sum_{\lambda=1}^{l-1} c_\lambda A^{(\lambda)}.$$

Then the statement holds for M as well. The construction of $b_{i_l,l}$ and $A^{(l)}$ did not depend on the choice of A. □

In the algorithm for finding all uniserial modules, Lemma 2.3 may be used to limit the search for generating elements.

As we are dealing with a p-group, the radical and socle series of $\mathbb{F}G$ coincide, or more precisely, $\mathrm{Rad}^{L-\lambda}(\mathbb{F}G) = \mathrm{Soc}^\lambda(\mathbb{F}G)$, where $L = \ell\ell(\mathbb{F}G)$. Thus, it is convenient to use Jennings' construction of a filtered basis, filtered with respect to powers of the radical.

A note on extending the procedure to group rings which have more than one simple module: finding a series of distinguished generating elements for a uniserial module and its submodules is a bit more cumbersome. One must deal with G-orbits of the generators when choosing b_{λ,i_λ}, among other details.

§3 An Example of the Procedure for Finding Uniserial Submodules of $\mathbb{F}G$

As a first example of the procedure, let

$$G = Q_8 = \langle x, y | x^4 = 1, x^2 = y^2,\ {}^y x = x^{-1} \rangle,$$

and let $\mathbb{F}$ be a field of characteristic 2. The Jennings series of subgroups is $\kappa_1 = Q_8 \supset \kappa_2 = \langle x^2 \rangle \supset \kappa_3 = \langle 1 \rangle$. By Jennings' theorem, we may choose generating sets $\{x_{1,1} = \bar{x}, x_{1,2} = \bar{y}\}$, and $\{x_{2,1} = \bar{x}^2\}$. Since $\bar{x}^2 = \bar{y}^2$, we may use $\bar{x}^2\bar{y}$ for a basis element, going against Jennings' specifications, and still be using the same elements.

We obtain the following filtered basis for $\mathbb{F}G$:

$$\begin{array}{ccc} & \beta_{0,1} = 1 & \\ \beta_{1,1} = \bar{y} & & \beta_{1,2} = \bar{x} \\ \beta_{2,1} = \bar{x}\bar{y} & & \beta_{2,2} = \bar{x}^2 \\ \beta_{3,1} = \bar{x}^2\bar{y} & & \beta_{3,2} = \bar{x}^3 \\ & \beta_{4,1} = \bar{x}^3\bar{y} & \end{array}$$

The notation we wish to use for our basis is $b_{\lambda,i} = \beta_{(5-\lambda),i}$, emphasizing powers of the socle, rather than powers of the radical.

The geometric presentation of the basis elements will be exploited to represent elements of $\mathbb{F}G$. A vector in $\mathbb{F}G$ is written with the shape given above, with the coefficient for each basis element in the corresponding location. This gives a clearer meaning than the standard linear way to explicitly express a vector.

From the previous section, we let $A^{(1)} = b_{1,1}$.

Since $\mathrm{Soc}^2(\mathbb{F}G)/\mathrm{Soc}(\mathbb{F}G)$ is semisimple, and $\mathrm{Rad}(\mathrm{Soc}^2(\mathbb{F}G))$ is one dimensional, any element of $\mathrm{Soc}^2(\mathbb{F}G) - \mathrm{Soc}(\mathbb{F}G)$ generates a 2-dimensional uniserial module. The set of distinguishing generators for 2-dimensional uniserial submodules is $\{A_r^{(2)} = b_{2,1} + r b_{2,2} \mid r \in \mathbb{F}\} \cup \{B^{(2)} = b_{2,2}\}$.

It is necessary to take into account the left action of $\mathbb{F}Q_8$. As x and y generate G, the radical of $\mathbb{F}G$ is generated by $\bar{x}$ and $\bar{y}$. If M is a module, $\mathrm{Rad}(M) = \mathrm{Rad}(\mathbb{F}G)M = \mathrm{Span}(\bar{x}M, \bar{y}M)$. We need to find the left actions of $\bar{x}$ and $\bar{y}$ on the basis elements.

The left action of $\bar{x}$ on the basis elements is clear. Remember that $\bar{x}^4 = (x-1)^4 = x^4 - 1 = 0$, as we are dealing with a field of characteristic 2. It may be calculated that $\bar{y}\bar{x} = \bar{x}\bar{y} + (\bar{x}^2 + \bar{x}^3)(\bar{y}+1)$, and that $\bar{y}$ commutes with $\bar{x}^2$ and $\bar{x}^3$. Also, $\bar{y}^2 = \bar{x}^2$. Thus $\bar{y}\ \bar{x}\bar{y} = \bar{x}^2\bar{y} + \bar{x}^3 + \bar{x}^3\bar{y}$.

In vector notation, as described above, we write an arbitrary element A of $\mathbb{F}Q_8$, as well as $\bar{x}A$, and $\bar{y}A$ as

$$A = \begin{array}{ccc} & a_5 & \\ a_{4,1} & & a_{4,2} \\ a_{3,1} & & a_{3,2} \\ a_{2,1} & & a_{2,2} \\ & a_1 & \end{array}, \qquad \bar{x}A = \begin{array}{ccc} & 0 & \\ 0 & & a_5 \\ a_{4,1} & & a_{4,2} \\ a_{3,1} & & a_{3,2} \\ & a_{2,1} & \end{array},$$

$$\bar{y}A = \begin{array}{ccc} & 0 & \\ a_5 & & 0 \\ a_{4,2} & & a_{4,1}+a_{4,2} \\ a_{4,2}+a_{3,1}+a_{3,2} & & a_{4,2}+a_{3,1} \\ & a_{4,2}+a_{3,1}+a_{2,2} & \end{array}$$

or, more conveniently,

$$A = \begin{array}{ccc} & a & \\ b & & c \\ d & & e \\ f & & g \\ & h & \end{array}, \qquad \bar{x}A = \begin{array}{ccc} & \cdot & \\ \cdot & & a \\ b & & c \\ d & & e \\ & f & \end{array}, \qquad \bar{y}A = \begin{array}{ccc} & \cdot & \\ a & & \cdot \\ c & & b+c \\ c+d+e & & c+d \\ & c+d+g & \end{array}$$

where the character "$\cdot$" deemphasizes the zero entries.

When multiplying on the right we obtain

$$A\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & a \\ b & & b+c \\ b+d & & b+d+e \\ & b+d+f & \end{array}, \text{ and } \quad A\bar{y} = \begin{array}{ccc} & \cdot & \\ a & & \cdot \\ c & & b \\ e & & d \\ & g & \end{array}$$

Using our vector notation, the distinguished generators for two-dimensional uniserial modules are:

$$A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & r \\ & \cdot & \end{array} \quad (r \in \mathbb{F}), \qquad B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}$$

Assume A is a distinguished generator for a uniserial module of length 3. We must have $a = b = c = 0$, since $A \in \mathrm{Soc}^3(\mathbb{F}G)$. These restrictions give

$$A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ d & & e \\ f & & g \\ & h & \end{array}, \qquad \bar{x}A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ d & & e \\ & f & \end{array}, \qquad \bar{y}A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ d+e & & d \\ & d+g & \end{array}$$

Either $\bar{x}A$ or $\bar{y}A$ must be an element of $\mathrm{Soc}^2(\mathbb{F}G) - \mathrm{Soc}(\mathbb{F}G)$, so d or e must be non-zero. This forces the coefficient of $b_{2,1}$ and $b_{2,2}$ to be non-zero in at least one of $\bar{x}A$ and $\bar{y}A$. As they must both be in the same module, both coefficients must be non-zero. We must have an extension of $\mathbb{F}GA_r^{(2)}$, where $r \neq 0$. By Lemma 2.3 parts ii) and iii), we need $d = 1$, which forces $e = r$, and $1 + r = 1/r$. This implies that r must be a primitive third root of unity, as $\mathrm{char}(\mathbb{F}) = 2$. By Lemma 2.3iv), we may assume that $f = h = 0$. We then have,

$$A_{r,s}^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ 1 & & r \\ \cdot & & s \\ & \cdot & \end{array},$$

$$\bar{x}A_{r,s}^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & r \\ & \cdot & \end{array} = A_r^{(2)}, \qquad \bar{y}A_{r,s}^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1+r & & 1 \\ & 1+s & \end{array} = \frac{1}{r}A_r^{(2)} + (1+s)A^{(1)}$$

where r is a primitive third root of unity and $s \in \mathbb{F}$.

Now we show that there is no uniserial module in $\mathbb{F}G$ of length 4.

Assume A is a distinguished generator for a uniserial module of length 4. By the multiplication shown above, we must have $a = 0$. The element $\bar{x}A$ must have a non-zero coefficient for either $b_{3,1}$ or $b_{3,2}$. Since this must be a generator of a uniserial module of length 3, both coefficients must be non-zero, so both b and c must be non-zero. Using Lemma 2.3 parts iii) and iv), we may assume that $b = 1$

and $d = f = h = 0$. These restrictions give

$$A = \begin{array}{ccc} & \cdot & \\ 1 & & c \\ \cdot & & e \\ \cdot & & g \\ & \cdot & \end{array}, \qquad \bar{x}A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ 1 & & c \\ \cdot & & e \\ & \cdot & \end{array}, \qquad \bar{y}A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ c & & 1+c \\ c+e & & c \\ & c+g & \end{array}$$

We must choose $c = r$ and $e = s$, giving

$$A = \begin{array}{ccc} & \cdot & \\ 1 & & r \\ \cdot & & s \\ \cdot & & g \\ & \cdot & \end{array}, \qquad \bar{x}A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ 1 & & r \\ \cdot & & s \\ & \cdot & \end{array}, \qquad \bar{y}A = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ r & & 1+r \\ r+s & & r \\ & r+g & \end{array}$$

We now have $\bar{x}A = A^{(3)}_{r,s}$. For $\bar{y}A$ to be in $\mathbb{F}GA^{(3)}_{r,s}$, we must have

$$\bar{y}A = rA^{(3)}_{r,s} + (r+s)A^{(2)}_{r} + (r+g)A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ r & & r^2 \\ r+s & & rs+r^2+rs \\ & r+g & \end{array}$$

In our field, $rs + r^2 + rs = r^2 = r + 1$. However, the coefficient for $b_{2,2}$ needs to be r, not $r + 1$, which leads to a contradiction. This leads to the following theorem.

Theorem 3.1. *The isomorphism classes of uniserial modules of $\mathbb{F}Q_8$ are in 1-to-1 correspondence with the distinguished generators*

$$\{A^{(1)}, A^{(2)}_{r}, B^{(2)}, A^{(3)}_{\omega,0} \mid r, \omega \in \mathbb{F}, \omega^2 + \omega + 1 = 0\}.$$

(If $\mathbb{F}$ does not contain a primitive third root of unity, then there is no uniserial module of length 3 for $\mathbb{F}Q_8$.)

Proof. By Lemma 1.5 we know that all uniserial left modules of $\mathbb{F}Q_8$ are isomorphic to submodules of $\mathbb{F}Q_8$, taken as a left module. By Lemma 2.3, and the work above, all uniserial submodules of $\mathbb{F}Q_8$ may be generated by one of the elements in the list $\{A^{(1)},\quad A_r^{(2)},\quad B^{(2)},\quad A_{\omega,s}^{(3)} \mid r,s,\omega\in\mathbb{F},\omega^2+\omega+1=0\}$. The action of $\mathbb{F}Q_8$ on a module characterizes the module.

There is only one isomorphism class of 1-dimensional modules for $\mathbb{F}Q_8$. They are thus isomorphic to $\mathbb{F}Q_8A^{(1)}$.

Assume M is a 2-dimensional uniserial module generated by A. If $\bar{x}A=0$, then $\bar{x}M=0$, and $M\cong\mathbb{F}GB^{(2)}$. If $\bar{x}A\neq 0$, then $\bar{y}A=r\bar{x}A$, for some $r\in\mathbb{F}$. This only occurs when $M\cong\mathbb{F}GA_r^{(2)}$.

Two modules M and N are isomorphic over $\mathbb{F}G$ if and only if the group representations of G with respect to M and N are isomorphic.

Assume M is a 3-dimensional uniserial module. For some $\omega,s\in\mathbb{F}$, $M=\mathbb{F}GA_{\omega,s}^{(3)}$. Let $A=A_{\omega,s}^{(3)}-(s+1)A_\omega^{(2)}$. As this element has a non-zero coefficient for $A_{\omega,s}^{(3)}$, it generates M. A basis for M is given by $\mathfrak{B}=\{A,\bar{x}A,\bar{x}^2A\}$, where $\bar{x}A=A_\omega^{(2)}-(s+1)A^{(1)}$, and $\bar{x}^2A=A^{(1)}$. The action of $\bar{y}$ on this basis is then $\bar{y}A=(1+\omega)A_\omega^{(2)}+(1+s)A^{(1)}-\omega(s+1)A^{(1)}=(1+\omega)\bar{x}A$, and $\bar{y}\;\bar{x}A=\omega\bar{x}^2A$. Thus, regardless of s, there is a basis for M such that the action of x and y are given by the matrices

$$\rho(x)=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix},\qquad \rho(y)=\begin{pmatrix}1&0&0\\1+\omega&1&0\\0&\omega&1\end{pmatrix}$$

as $x=\bar{x}+1$, $y=\bar{y}+1$. Because $\operatorname{Rad}(M)=\mathbb{F}Q_8A_\omega^{(2)}\not\cong\mathbb{F}Q_8A_{\omega+1}^{(2)}$, there are exactly two isomorphism classes of uniserial modules of length 3 for $\mathbb{F}Q_8$. □

We now present an alternative proof of Theorem 3.1 which uses a more practical method. We use this second method to find the isomorphism classes of uniserial modules in the next chapter.

Proof. By Lemma 1.5 we know that all uniserial left modules of $\mathbb{F}Q_8$ are isomorphic to submodules of $\mathbb{F}Q_8$, taken as a left module. By Lemma 2.3, and the work above, all uniserial submodules of $\mathbb{F}Q_8$ may be generated by one of the elements in the list $\{A^{(1)},A_r^{(2)},B^{(2)},A_{\omega,s}^{(3)} \mid r,s,\omega\in\mathbb{F},\omega^2+\omega+1=0\}$. Let A be one of these generators, and let $M=\mathbb{F}Q_8A$. By Proposition 1.8 we have $\operatorname{Hom}(\mathbb{F}Q_8A,\mathbb{F}Q_8)\cong A\mathbb{F}Q_8$, so the set of homomorphisms is spanned by the set $\{A\beta_{\lambda,i}\}$. For each of these distinguished elements we have,

$$
\begin{array}{lll}
A1 = A, & A^{(n)}\beta_{\lambda,i} = 0 & (\lambda > n) \\
A_r^{(2)}\bar{x} = A^{(1)}, & A_r^{(2)}\bar{y} = rA^{(1)} & \\
B^{(2)}\bar{x} = 0, & B^{(2)}\bar{y} = A^{(1)} & \\
A_{\omega,s}^{(3)}\bar{x} = A_{\omega^2}^{(2)} + A^{(1)}, & A_{\omega,s}^{(3)}\bar{y} = \omega A_{\omega^2}^{(2)} + sA^{(1)}, & \\
\quad A_{\omega,s}^{(3)}\bar{x}^2 = A^{(1)}, & \quad A_{\omega,s}^{(3)}\bar{x}\bar{y} = \omega A^{(1)} &
\end{array}
$$

From this information we may list all homomorphic images of each uniserial module.

There is only one isomorphism class of 1-dimensional modules for $\mathbb{F}Q_8$. They are thus isomorphic to $\mathbb{F}Q_8A^{(1)}$.

Let $M_r^{(2)} = \mathbb{F}GA_r^{(2)}$. The homomorphic images of $A_r^{(2)}$ are of the form $\alpha A_r^{(2)} + \beta A^{(1)}$, where α and β are elements of $\mathbb{F}$. All such elements are elements of $M_r^{(2)}$. The isomorphisms occur precisely when $\alpha \neq 0$. Thus, for each $r \in \mathbb{F}$, the module $M_r^{(2)}$ forms its own isomorphism class. Likewise, $N^{(2)} = \mathbb{F}GB^{(2)}$ forms its own isomorphism class.

Let $M_{\omega,s}^{(3)} = \mathbb{F}GA_{\omega,s}^{(3)}$. Given an $\mathbb{F}G$-homomorphism ϕ of M, we wish to find the distinguished generating element of $\phi(M)$. We know that $\phi(A(3)_{\omega,s})$ is of the form $\alpha A_{\omega,s}^{(3)} + \beta A_{\omega^2}^{(2)} + \gamma A^{(1)}$. Writing this in our vector notation, this is

$$
\phi\begin{pmatrix} & \cdot & \\ \cdot & & \cdot \\ 1 & & \omega \\ \cdot & & s \\ & \cdot & \end{pmatrix} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \alpha & & \alpha\omega \\ \beta & & \alpha s + \beta\omega^2 \\ & \gamma & \end{array}
$$

If $\alpha = 0$, then ϕ is not an isomorphism. By using different values for β and γ we see that $M_{\omega^2}^{(2)}$ is a homomorphic image of $M_{\omega,s}^{(3)}$, as is $M^{(1)} = \mathbb{F}GA^{(1)}$.

If $\alpha \neq 0$, the image of $M_{\omega,s}^{(3)}$ is 3-dimensional, so ϕ must be an isomorphism. We find the elements $\bar{x}^2/\alpha\phi(A_{\omega,s}^{(3)}) = A^{(1)}$ and $\big(\bar{x}\phi(A_{\omega,s}^{(3)}) - \beta A^{(1)}\big)/\alpha = A_\omega^{(2)}$ are elements in the image of ϕ. Thus the element

$$
\begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \alpha & & \alpha\omega \\ \beta & & \alpha s + \beta\omega^2 \\ & \gamma & \end{array}
-
\begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \beta & & \beta\omega \\ & \gamma & \end{array}
=
\begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \alpha & & \alpha\omega \\ \cdot & & \alpha s + \beta \\ & \cdot & \end{array}
$$

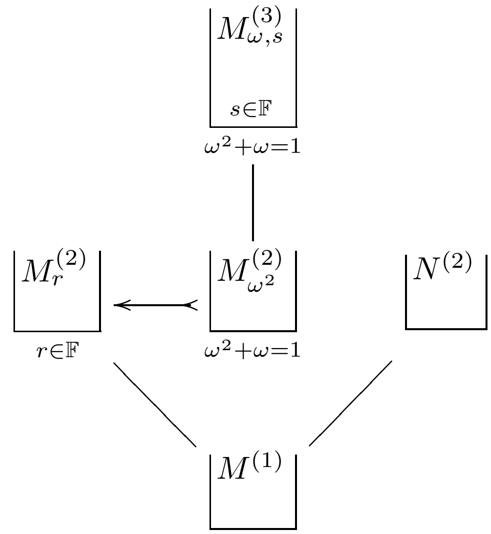
is also an element of $\phi(M_{\omega,s}^{(3)})$.

If we look at all possible isomorphisms of $M^{(3)}_{\omega,s}$, we see that setting $\alpha = 1$ and letting β vary over $\mathbb{F}$ that $M^{(3)}_{\omega,s} \cong M^{(3)}_{\omega,s'}$ for all $s' \in \mathbb{F}$. We may choose the module $M^{(3)}_{\omega,0}$ to be the representative of this isomorphism class. $\square$

We now display two lattices of all the uniserial submodules of $\mathbb{F}Q_8$. The first lattice shows the sub-module structure of the uniserial modules. A submodule of another module is listed below it, and there is a vertical or diagonal connection between them. The horizontal arrow indicates that the module $M^{(2)}_{\omega}$ is a module of the type $M^{(2)}_{r}$, where $r \in \mathbb{F}$. We define the modules as we did in the proof.

$$
\begin{tikzcd}
 & M^{(3)}_{\omega,s} \arrow[d, dash] & \\
M^{(2)}_{r} \arrow[dr, dash] & M^{(2)}_{\omega} \arrow[l, tail] & N^{(2)} \arrow[dl, dash] \\
 & M^{(1)} &
\end{tikzcd}
$$

The second lattice describes the homomorphism structure of the modules. If M is a homomorphic image of N then the module M appears at a lower level in the lattice than the module N, and there is a vertical or diagonal line connecting them. The horizontal arrow indicates that the module $M^{(2)}_{\omega^2}$ is a module of the type $M^{(2)}_{r}$, where $r \in \mathbb{F}$. Each isomorphism class is represented by a "bucket". The information inside the bucket describes all modules that lie in the same isomorphism class. The information outside the bucket describes all isomorphism classes of this type. Thus the top bucket implies that $M^{(3)}_{\omega,s} \cong M^{(3)}_{\omega',s'}$ whenever s and s' are elements of $\mathbb{F}$, and only when $\omega = \omega'$. These modules exist for all $\omega \in \mathbb{F}$ such that $\omega^2 + \omega = 1$.



In the next chapter we use similar notation. For simplicity, we place the name of the generator of a uniserial module in the bucket, rather than the name of the module. In doing this, we do not need to name all the modules, and there is less possibility of confusing the generators of the modules.

Chapter 2: Classification of Uniserial Modules for Groups of Order 16

In this chapter we explicitly describe the uniserial $\mathbb{F}G$-modules, where $\mathbb{F}$ is a field of characteristic 2, and G is one of a certain collection of 2-groups. We restrict ourselves to the groups of order 16 that are non-abelian, and may not be written as the direct product of two non-trivial groups. There are seven such groups ([H&S, Sc, As]), and we denote these G_1, $(2 \times 4).2$, Mod_{16}, D_{16}, SD_{16}, Q_{16}, and $D_8\mathrm{Y}4$. Each group will be considered in its own section.

Each section concentrates on one specific group, G. In each section the presentation of material follows this pattern:

- A presentation of G is given.
- We describe the partially ordered set of all uniserial submodules of the regular representation $\mathbb{F}G$, were the ordering is by inclusion.
- We describe the partially ordered set of all isomorphism classes of $\mathbb{F}G$ modules, where $M \leq N$ indicates that M is a homomorphic image of N.

The modules are then defined explicitly in the following way:

- A basis is given for the regular representation $\mathbb{F}G$.
- An arbitrary element A of $\mathbb{F}G$ is given.
- We give actions of generators of $\mathbb{F}G$ on A.
- We list the uniserial submodules U of $\mathbb{F}G$ in increasing order of dimension. We do this for each uniserial submodule by listing a distinguished generator B of U, along with the left and right action of generators of $\mathbb{F}G$ on B.

Each module is denoted by listing its distinguished generator. These generators are written as symbols such as $A^{(3)}_{r,s}$. Here the letter A denotes the family to which the module belongs, the superscript denotes the dimension of the module generated by this element, and the subscripts are parameters which lie in $\mathbb{F}$. If different values are given to these subscripts, then different modules are represented. The particular definition of each generator is given after the diagrams of inclusion and homomorphism.

In the diagram that represents the partially ordered set of uniserial modules, if for example, $A^{(3)}_{r,s}$ is above $A^{(2)}_{r}$ with a solid line connecting them, this indicates that, for any fixed $r \in \mathbb{F}$ and all $s \in \mathbb{F}$, the module generated by $A^{(3)}_{r,s}$ contains the module generated by $A^{(2)}_{r}$. In the diagrams, if there is an arrow from, let us say,

$A_1^{(2)}$ to $A_r^{(2)}$, this indicates that the module generated by $A_1^{(2)}$ is a specific case of the more generally described module generated by $A_r^{(2)}$. A dotted line between $C_{s,t}^{(3)}$ and $A_{r,s}^{(3)}$ indicates that there is overlap in the definitions of the modules generated by these generators. The overlap is described with the dotted lines.

In the homomorphism diagram, isomorphism classes are represented by *buckets*. Each bucket contains a list of all items which are in a particular isomorphism class. We describe the general bucket by analogy with the description of the following specific bucket:

$$\begin{array}{c} \begin{array}{|c|} A_{r,s,t}^{(4)} \\ \\ {\scriptstyle s\in\mathbb{F}} \\ \hline \end{array} \\ r \neq 1 \end{array}$$

All information pertaining to a specific isomorphism class is contained inside of the bucket. All information listed outside of the bucket is used to distinguish different isomorphism classes, or to determine permitted values of variables. Given $r, s, t \in \mathbb{F}$, where $r \neq 1$, our sample bucket represents the isomorphism class of $\mathbb{F}GA_{r,s,t}^{(4)}$. The variable s is permitted to vary over all the elements of $\mathbb{F}$. Thus if $r \neq 1$, then $\mathbb{F}GA_{r,s,t}^{(4)} \cong \mathbb{F}GA_{\rho,\sigma,\tau}^{(4)}$ if and only if $\rho = r$ and $\tau = t$, where all parameters are elements of $\mathbb{F}$.

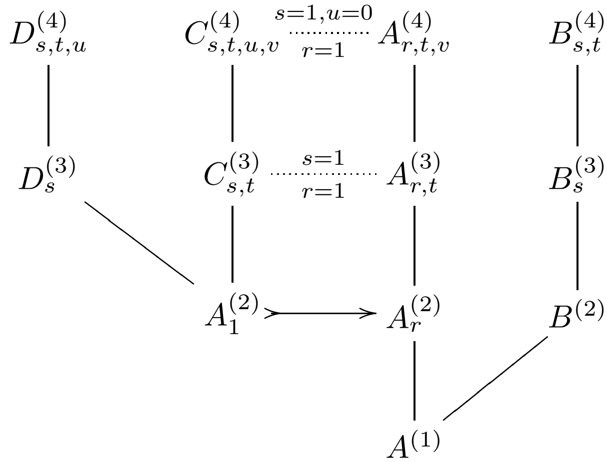
Assume, for example, that in a diagram the bucket containing $A_{r,s,t}^{(4)}$ is above another bucket containing $C_{r,s}^{(3)}$. This indicates that there is a surjective homomorphism from the module generated by $A_{r,s,t}^{(4)}$ to the module generated by $C_{r,s}^{(3)}$. It must be noted that r and s have the same values in each of these generators.

The homomorphism diagram lists all situations where there is a homomorphism from one uniserial module onto another uniserial module. If two modules are isomorphic, then their distinguished generators are listed in the same bucket. If there is a module homomorphism $\phi : M \to N$, then the bucket containing the distinguished generator of M is above the bucket containing the distinguished generator of N, and there is a line or series of lines connecting them. There may also be parameters in each bucket, and these must match.

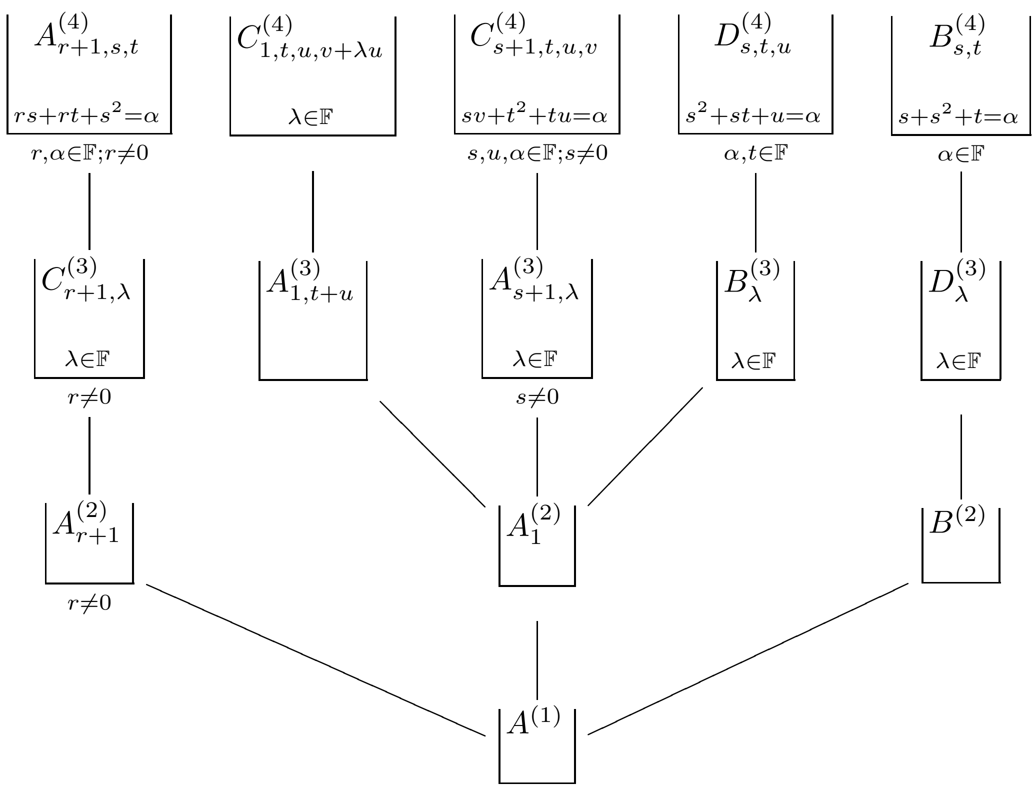
§1 Uniserial submodules of $\mathbb{F}G_1$

Group: $G_1 = \langle x, y \mid x^4 = y^4 = 1, yx = x^{-1}y^{-1}, \{x^2, y^2\} \subset Z \rangle$, where Z is the center of the group.

This is the partially ordered set of uniserial $\mathbb{F}G_1$ submodules:



This is the homomorphism diagram for uniserial $\mathbb{F}G_1$ modules:



A basis for $\mathbb{F}G_1$ is:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \bar{x} & & \bar{y} & & \\
 & \bar{x}^2 & & \bar{x}\bar{y} & & \bar{y}^2 & \\
\bar{x}^3 & & \bar{x}^2\bar{y} & & \bar{x}\bar{y}^2 & & \bar{y}^3 \\
 & \bar{x}^3\bar{y} & & \bar{x}^2\bar{y}^2 & & \bar{x}\bar{y}^3 & \\
 & & \bar{x}^3\bar{y}^2 & & \bar{x}^2\bar{y}^3 & & \\
 & & & \bar{x}^3\bar{y}^3 & & &
\end{array}$$

We denote an arbitrary element of $\mathbb{F}G_1$ by specifying its coefficients as follows:

$$A = \begin{array}{ccccccc}
 & & & a & & & \\
 & & b & & c & & \\
 & d & & e & & f & \\
g & & h & & i & & j \\
 & k & & l & & m & \\
 & & n & & p & & \\
 & & & q & & &
\end{array}$$

The left and right action of the generators $\bar{x}$ and $\bar{y}$ on such an element are:

$$\bar{x}A = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & a & & \cdot & & \\
 & b & & c & & \cdot & \\
d & & e & & f & & \cdot \\
 & h & & i & & j & \\
 & & l & & m & & \\
 & & & p & & &
\end{array}$$

$$\bar{y}A = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & a & & \\
 & b & & b & & b+c & \\
b & & b+d+e & & b+e & & b+e+f \\
 & b+e+g & & b+e+g+h+i & & b+e+i & \\
 & & b+e+g+i+k & & b+e+g+i+k+l+m & & \\
 & & & b+e+g+i+k+m+n & & &
\end{array}$$

$$A\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & a & & \cdot & & \\
 & b+c & & c & & c & \\
c+d+e & & c+e & & c+e+f & & c \\
 & c+e+h & & c+e+h+i+j & & c+e+j & \\
 & & c+e+h+j+k+l+m & & c+e+h+j+m & & \\
 & & & c+e+h+j+k+m+p & & &
\end{array}$$

$$A\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & a & & \\
 & \cdot & & b & & c & \\
\cdot & & d & & e & & f \\
 & g & & h & & i & \\
 & & k & & l & & \\
 & & & n & & &
\end{array}$$

The uniserial module of length 1:

$$A^{(1)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & \cdot & & \\
 & & & 1 & & &
\end{array}, \qquad \bar{x}A^{(1)} = \bar{y}A^{(1)} = A^{(1)}\bar{x} = A^{(1)}\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & \cdot & & \\
 & & & \cdot & & &
\end{array}$$

$\|G_1\| = A^{(1)}$

The uniserial modules of length 2:

$$A_r^{(2)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & r & & 1 & & \\
 & & & \cdot & & &
\end{array}, \quad \bar{x}A_r^{(2)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & \cdot & & \\
 & & & 1 & & &
\end{array}, \quad \bar{y}A_r^{(2)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & \cdot & & \\
 & & & r & & &
\end{array}$$

$\mathbb{F}G_1\|\langle x^2, y^2, xy\rangle\| = \mathbb{F}G_1 A_1^{(2)}, \qquad \|\langle y, x^2\rangle\| = A_0^{(2)}$

$$B^{(2)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & 1 & & \cdot & & \\
 & & & \cdot & & &
\end{array}, \quad \bar{x}B^{(2)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & \cdot & & \\
 & & & \cdot & & &
\end{array}, \quad \bar{y}B^{(2)} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & \cdot & & \\
 & & & 1 & & &
\end{array}$$

$\mathbb{F}G_1\|\langle x, y^2\rangle\| = \mathbb{F}G_1 B^{(2)}$

The uniserial modules of length 3:

$$A^{(3)}_{r,s} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &r&&r&&1& \\ &&s&&\cdot&& \\ &&&\cdot&&& \end{array}, \qquad \bar{x}A^{(3)}_{r,s} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&r&&1&& \\ &&&\cdot&&& \end{array}, \qquad \bar{y}A^{(3)}_{r,s} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&r&&1&& \\ &&&r+1+s&&& \end{array}$$

$$A^{(3)}_{r,s}\bar{x} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&1&&1&& \\ &&&r+1&&& \end{array}, \qquad A^{(3)}_{r,s}\bar{y} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&r&&r&& \\ &&&s&&& \end{array}$$

Homomorphic images: $\mathbb{F}G_1A^3_{r,s}/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1A^2_1$.
Isomorphisms: If $r \neq 1$ then $\mathbb{F}G_1A^3_{r,s} \cong \mathbb{F}G_1A^3_{r,0}, \forall s \in \mathbb{F}$. $\mathbb{F}G_1A^3_{1,s}$ forms its own isomorphism class.

$$C^{(3)}_{s,t} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &s&&1&&1& \\ &&t&&\cdot&& \\ &&&\cdot&&& \end{array}, \qquad \bar{x}C^{(3)}_{s,t} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&1&&1&& \\ &&&\cdot&&& \end{array}, \qquad \bar{y}C^{(3)}_{s,t} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&s&&s&& \\ &&&s+1+t&&& \end{array}$$

$$C^{(3)}_{s,t}\bar{x} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&s&&1&& \\ &&&s+1&&& \end{array}, \qquad C^{(3)}_{s,t}\bar{y} = \begin{array}{ccccccc} &&&\cdot&&& \\ &&\cdot&&\cdot&& \\ &\cdot&&\cdot&&\cdot& \\ \cdot&&\cdot&&\cdot&&\cdot \\ &\cdot&&\cdot&&\cdot& \\ &&s&&1&& \\ &&&t&&& \end{array}$$

Note: $C^{(3)}_{1,s} = A^{(3)}_{1,s}$.
Homomorphic images: $\mathbb{F}G_1C^3_{s,t}/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1A^2_s$.

Isomorphisms: If $s \neq 1$ then $\mathbb{F}G_1C^3_{s,t} \cong \mathbb{F}G_1C^3_{s,0}$. $\mathbb{F}G_1C^3_{1,t}$ has been mentioned above.

$$D_s^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & 1 & & \cdot & & \cdot & \\ & & \cdot & & s & & \\ & & & \cdot & & & \end{array}, \quad \bar{x}D_s^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & s & & & \end{array}, \quad \bar{y}D_s^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{array}$$

$$D_s^{(3)}\bar{x} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & \cdot & & \\ & & & 1+s & & & \end{array}, \qquad D_s^{(3)}\bar{y} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & \cdot & & \\ & & & \cdot & & & \end{array}$$

Homomorphic images: $\mathbb{F}G_1D^3_s/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1B^2$.
Isomorphisms: $\mathbb{F}G_1D^3_s \cong \mathbb{F}G_1D^3_0$

$$B_s^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & 1 & & 1 & & \cdot & \\ & & \cdot & & s & & \\ & & & \cdot & & & \end{array}, \quad \bar{x}B_s^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & \cdot & & \\ & & & s & & & \end{array}, \quad \bar{y}B_s^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & \cdot & & \\ & & & 1 & & & \end{array}$$

$$B_s^{(3)}\bar{x} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & 1+s & & & \end{array}, \qquad B_s^{(3)}\bar{y} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & 1 & & \\ & & & \cdot & & & \end{array}$$

Homomorphic images: $\mathbb{F}G_1B^3_s/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1A^2_1$.

Isomorphisms: $\mathbb{F}G_1B^3_s \cong \mathbb{F}G_1B^3_0$

The uniserial modules of length 4:

$$A^{(4)}_{r,s,t} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ r^2 & & r & & r & & 1 \\ & r^2+r & & s & & \cdot & \\ & & t & & \cdot & & \\ & & & \cdot & & & \end{array}, \qquad \bar{x}A^{(4)}_{r,s,t} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & r & & r & & 1 & \\ & & s & & \cdot & & \\ & & & \cdot & & & \end{array}, \qquad \bar{y}A^{(4)}_{r,s,t} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & r^2 & & r^2 & & r & \\ & & \cdot & & s & & \\ & & & t & & & \end{array}$$

$$A^{(4)}_{r,s,t}\bar{x} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & r & & 1 & & 1 & \\ & & 1+r^2+s & & r+1 & & \\ & & & 1+r^2 & & & \end{array}, \qquad A^{(4)}_{r,s,t}\bar{y} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & r^2 & & r & & r & \\ & & r^2+r & & s & & \\ & & & t & & & \end{array}$$

Homomorphic images: $\mathbb{F}G_1A^{(4)}_{r,s,t}/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1C^{(3)}_{r,r^2+r+s}$.
Isomorphisms: If $r \neq 1$, then $\mathbb{F}G_1A^{(4)}_{r,s,t} \cong \mathbb{F}G_1A^{(4)}_{r,0,s+t+s^2/(r+1)}$. $\mathbb{F}G_1A^{(4)}_{1,s,t}$ forms its own isomorphism class.
$\mathbb{F}G_1\|\langle x^{-1}y, x^2y^2\rangle\| = \mathbb{F}G_1A^{(4)}_{1,0,0} = \mathbb{F}G_1C^{(4)}_{1,0,0,0}$
$\mathbb{F}G_1\|\langle xy, x^2y^2\rangle\| = \mathbb{F}G_1A^{(4)}_{1,1,1} = \mathbb{F}G_1C^{(4)}_{1,1,0,1}$
$\|\langle y\rangle\| = A^{(4)}_{0,0,0}, \qquad \mathbb{F}G_1\|\langle x^2y\rangle\| = \mathbb{F}G_1A^{(4)}_{0,1,0}$

$$C^{(4)}_{s,t,u,v} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ s & & s & & 1 & & 1 \\ & u & & t & & \cdot & \\ & & v & & \cdot & & \\ & & & \cdot & & & \end{array}, \qquad \bar{x}C^{(4)}_{s,t,u,v} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & s & & 1 & & 1 & \\ & & t & & \cdot & & \\ & & & \cdot & & & \end{array},$$

$$\bar{y}C^{(4)}_{s,t,u,v} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & s & & 1 & & 1 & \\ & & s+1+u & & s+1+u+t & & \\ & & & s+1+u+v & & & \end{array}$$

$$C^{(4)}_{s,t,u,v}\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & s & & s & & 1 & \\
 & & s+1+u+t & & s+1 & & \\
 & & & s+1+u & & &
\end{array}, \qquad C^{(4)}_{s,t,u,v}\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & s & & s & & 1 & \\
 & & u & & t & & \\
 & & & v & & &
\end{array}$$

Note: $C^{(4)}_{1,t,0,v} = A^{(4)}_{1,t,v}$

Homomorphic images: $\mathbb{F}G_1C^{(4)}_{s,t,u,v}/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1A^{(3)}_{s,1+u+s^2+t} \cong \mathbb{F}G_1A^{(3)}_{s,t+u}$.

Isomorphisms: For any $\lambda \in \mathbb{F}$, $\mathbb{F}G_1C^{(4)}_{s+1,t,u,v} \cong \mathbb{F}G_1C^{(4)}_{s+1,t+\lambda s,u,v+\lambda u+\lambda^2 s}$. If $s = 0$, this simplifies to $\mathbb{F}G_1C^{(4)}_{s+1,t,u,v} = \mathbb{F}G_1C^{(4)}_{1,t,u,v} \cong \mathbb{F}G_1C^{(4)}_{1,t,u,v+\lambda u}$ for all $\lambda \in \mathbb{F}$.

$\mathbb{F}G_1\|\langle x^{-1}y, y^2\rangle\| = \mathbb{F}G_1C^{(4)}_{0,0,0,0}$, $\qquad \mathbb{F}G_1\|\langle xy, y^2\rangle\| = \mathbb{F}G_1C^{(4)}_{0,1,0,1}$

$$D^{(4)}_{s,t,u} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
1 & & 1 & & \cdot & & \cdot \\
 & \cdot & & s{+}t & & t & \\
 & & \cdot & & u & & \\
 & & & \cdot & & &
\end{array} \qquad \bar{x}D^{(4)}_{s,t,u} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & \cdot & & \cdot & \\
 & & s{+}t & & t & & \\
 & & & u & & &
\end{array} \qquad \bar{y}D^{(4)}_{s,t,u} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & \cdot & & \cdot & \\
 & & 1 & & 1{+}s & & \\
 & & & 1{+}t & & &
\end{array}$$

$$D^{(4)}_{s,t,u}\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & 1 & & \cdot & \\
 & & 1+s & & 1+t & & \\
 & & & 1+t+u & & &
\end{array}, \qquad D^{(4)}_{s,t,u}\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & 1 & & \cdot & \\
 & & \cdot & & s+t & & \\
 & & & \cdot & & &
\end{array}$$

Homomorphic images: $\mathbb{F}G_1D^{(4)}_{s,t,u}/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1B^{(3)}_{s+t} \cong \mathbb{F}G_1B^{(3)}_0$.

Isomorphisms: For all $\lambda \in \mathbb{F}$, $\mathbb{F}G_1D^{(4)}_{s,t,u} \cong \mathbb{F}G_1D^{(4)}_{\lambda+s,t,u+\lambda t+\lambda^2}$.

$\mathbb{F}G_1\|\langle xy, x^2\rangle\| = \mathbb{F}G_1D^{(4)}_{0,0,0}$, $\qquad \mathbb{F}G_1\|\langle xy^{-1}, x^2\rangle\| = \mathbb{F}G_1D^{(4)}_{1,0,1}$

$$
B^{(4)}_{s,t} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
1 & & \cdot & & \cdot & & \cdot \\
 & \cdot & & 1+s & & \cdot & \\
 & & \cdot & & t & & \\
 & & & \cdot & & &
\end{array}, \qquad
\bar{x}B^{(4)}_{s,t} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & 1+s & & \cdot & & \\
 & & & t & & &
\end{array}, \qquad
\bar{y}B^{(4)}_{s,t} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & 1 & & \cdot & \\
 & & 1 & & s & & \\
 & & & 1 & & &
\end{array}
$$

$$
B^{(4)}_{s,t}\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & 1+s & & \cdot & & \\
 & & & t & & &
\end{array}, \qquad
B^{(4)}_{s,t}\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & \cdot & & \cdot & \\
 & & \cdot & & 1+s & & \\
 & & & \cdot & & &
\end{array}
$$

Homomorphic images: $\mathbb{F}G_1B^{(4)}_{s,t}/\mathbb{F}G_1A^1 \cong \mathbb{F}G_1D^{(3)}_{1+s} \cong \mathbb{F}G_1D^{(3)}_0$.
Isomorphisms: for any $\lambda \in \mathbb{F}$, $\mathbb{F}G_1B^{(4)}_{s,t} \cong \mathbb{F}G_1B^{(4)}_{s+\lambda,t+\lambda+\lambda^2} \cong \mathbb{F}G_1B^{(4)}_{0,s+s^2+t}$.
$\|\langle x\rangle\| = B^{(4)}_{1,0}, \qquad \mathbb{F}G_1\|\langle xy^2\rangle\| = \mathbb{F}G_1B^{(4)}_{0,0}$

There are no uniserial modules of length 5.

§2 Uniserial submodules of $\mathbb{F}(2\times 4).2$

Group: $(2\times 4).2 = \langle x, y \mid x^4 = y^4 = 1,\ {}^{x}y = y^{-1}\rangle$

This is the partially ordered set of uniserial $\mathbb{F}(2\times 4).2$ submodules:

$$
\begin{tikzcd}[arrows=dash]
A^{(4)}_{r,s,t} \arrow[d] & & B^{(4)}_{s,t} \arrow[d] \\
A^{(3)}_{r,s} \arrow[d] & & B^{(3)}_{s} \arrow[d] \\
A^{(2)}_{r} \arrow[dr] & & B^{(2)} \arrow[dl] \\
& A^{(1)} &
\end{tikzcd}
$$

This is the homomorphism diagram for uniserial $\mathbb{F}(2\times 4).2$ modules:

$$
\begin{tikzcd}[arrows=dash]
\underset{\lambda\in\mathbb{F}}{A^{(4)}_{r,s+\lambda,t+\lambda(1+r+\lambda)}} \arrow[d] & & B^{(4)}_{s,t} \arrow[d] \\
\underset{\lambda\in\mathbb{F}}{A^{(3)}_{r+1,\lambda}} \arrow[d] & & B^{(3)}_{s} \arrow[d] \\
A^{(2)}_{r} \arrow[dr] & & B^{(2)} \arrow[dl] \\
& A^{(1)} &
\end{tikzcd}
$$

A basis for $\mathbb{F}(2 \times 4).2$ is:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \bar{x} & & \bar{y} & & \\
 & \bar{x}^2 & & \bar{x}\bar{y} & & \bar{y}^2 & \\
\bar{x}^3 & & \bar{x}^2\bar{y} & & \bar{x}\bar{y}^2 & & \bar{y}^3 \\
 & \bar{x}^3\bar{y} & & \bar{x}^2\bar{y}^2 & & \bar{x}\bar{y}^3 & \\
 & & \bar{x}^3\bar{y}^2 & & \bar{x}^2\bar{y}^3 & & \\
 & & & \bar{x}^3\bar{y}^3 & & &
\end{array}
$$

$$
A = \begin{array}{ccccccc}
 & & & a & & & \\
 & & b & & c & & \\
 & d & & e & & f & \\
g & & h & & i & & j \\
 & k & & l & & m & \\
 & & n & & p & & \\
 & & & q & & &
\end{array}, \qquad
\bar{x}A = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & a & & \cdot & & \\
 & b & & c & & \cdot & \\
d & & e & & f & & \cdot \\
 & h & & i & & j & \\
 & & l & & m & & \\
 & & & p & & &
\end{array},
$$

$$
\bar{y}A = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & a & & \\
 & \cdot & & b & & b+c & \\
\cdot & & d & & b+e & & b+e+f \\
 & g & & g+h & & b+e+i & \\
 & & g+k & & g+k+l & & \\
 & & & g+k+n & & &
\end{array}
$$

$$
A\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & a & & \cdot & & \\
 & b & & c & & c & \\
d & & e & & c+e+f & & c \\
 & h & & e+h+i & & c+e+j & \\
 & & h+k+l & & e+h+m & & \\
 & & & h+k+p & & &
\end{array}
$$

$$
A\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & a & & \\
 & \cdot & & b & & c & \\
\cdot & & d & & e & & f \\
 & g & & h & & i & \\
 & & k & & l & & \\
 & & & n & & &
\end{array}
$$

The uniserial module of length 1:

$$A^{(1)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & 1 & & & \end{array}, \qquad \bar{x}A^{(1)} = \bar{y}A^{(1)} = A^{(1)}\bar{x} = A^{(1)}\bar{y} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{array}$$

$\|(2x4).2\| = A^{(1)}$

The uniserial modules of length 2:

$$A_r^{(2)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & 1 & & r & & \\ & & & \cdot & & & \end{array}, \qquad \bar{x}A_r^{(2)} = A_r^{(2)}\bar{x} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & r & & & \end{array}, \qquad \bar{y}A_r^{(2)} = A_r^{(2)}\bar{y} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & 1 & & & \end{array}$$

$\|\langle x, y^2\rangle\| = A_0^{(2)}, \qquad \|\langle xy, y^2\rangle\| \in \mathbb{F}(2\times 4).2A_1^{(2)}$

$$B^{(2)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & 1 & & \\ & & & \cdot & & & \end{array}, \qquad \bar{x}B^{(2)} = B^{(2)}\bar{x} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & 1 & & & \end{array}, \qquad \bar{y}B^{(2)} = B^{(2)}\bar{y} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{array}$$

$\|\langle y, x^2\rangle\| = B^{(2)}$

The uniserial modules of length 3:

$$A_{r,s}^{(3)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & & \cdot \\ & 1 & & r+1 & & r^2+r & \\ & & \cdot & & s & & \\ & & & \cdot & & & \end{array},$$

$$\bar{x}A^{(3)}_{r,s} = \quad r+1 \quad r^2+r \quad s \quad ,\qquad \bar{y}A^{(3)}_{r,s} = \quad 1 \quad r \quad 1$$

$$A^{(3)}_{r,s}\bar{x} = \quad r \quad r^2+r \quad s+1 \quad ,\qquad A^{(3)}_{r,s}\bar{y} = \quad 1 \quad r+1$$

$$B^{(3)}_s = \quad 1 \quad s \quad ,\qquad \bar{x}B^{(3)}_s = B^{(3)}_s\bar{x} = \quad 1 \quad ,\qquad \bar{y}B^{(3)}_s = B^{(3)}_s\bar{y} = \quad s$$

The uniserial modules of length 4:

$$A^{(4)}_{r,s,t} = \quad 1 \quad r \quad r^2+r \quad r^3+r^2 \quad 1+r+s \quad r^2+r \quad t \quad ,$$

$$\bar{x}A^{(4)}_{r,s,t} = \quad r \quad r^2{+}r \quad r^3{+}r^2 \quad 1{+}r{+}s \quad r^2{+}r \quad t \quad ,\qquad \bar{y}A^{(4)}_{r,s,t} = \quad 1 \quad 1{+}r \quad r^2{+}r \quad 1 \quad r{+}s$$

$$
A^{(4)}_{r,s,t}\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & r & & r^2 & & r^3+r^2 & \\
 & & s+1 & & r^2 & & \\
 & & & r+t & & &
\end{array}, \qquad
A^{(4)}_{r,s,t}\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & 1 & & r & & r^2+r & \\
 & & \cdot & & r+s+1 & & \\
 & & & \cdot & & &
\end{array}
$$

$\|\langle x\rangle\| \in \mathbb{F}(2\times 4).2A^{(4)}_{0,1,0} \cong \mathbb{F}(2\times 4).2A^{(4)}_{0,0,0} \ni \|\langle xy^2\rangle\|$
$\|\langle xy\rangle\| \in \mathbb{F}(2\times 4).2A^{(4)}_{1,0,0} \cong \mathbb{F}(2\times 4).2A^{(4)}_{1,1,1} \ni \|\langle xy^{-1}\rangle\|$

$$
B^{(4)}_{s,t} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & 1 \\
 & \cdot & & s & & \cdot & \\
 & & t & & \cdot & & \\
 & & & \cdot & & &
\end{array}, \qquad
\bar{x}B^{(4)}_{s,t} = B^{(4)}_{s,t}\bar{x} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & 1 & \\
 & & s & & \cdot & & \\
 & & & \cdot & & &
\end{array}, \qquad
\bar{y}B^{(4)}_{s,t} = B^{(4)}_{s,t}\bar{y} = \begin{array}{ccccccc}
 & & & \cdot & & & \\
 & & \cdot & & \cdot & & \\
 & \cdot & & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & & \cdot & \\
 & & \cdot & & s & & \\
 & & & t & & &
\end{array}
$$

$\|\langle y\rangle\| = B^{(4)}_{0,0}, \qquad \|\langle x^2y\rangle\| = B^{(4)}_{1,0}$

There are no uniserial modules of length 5.

§3 Uniserial submodules of $\mathbb{F}\mathbf{Mod}_{16}$

Group: $\mathrm{Mod}_{16} = \langle x, y \mid x^8 = y^2 = 1,\ {}^{y}x = x^5 \rangle$

This is the partially ordered set of uniserial $\mathbb{F}\mathrm{Mod}_{16}$ submodules:

$$
\begin{tikzcd}
B^{(8)}_{u,v,w,x} \arrow[d, dash] & & \\
B^{(7)}_{0,u,v,w} \arrow[rr, tail] & & B^{(7)}_{\delta,u,v,w} \arrow[d, dash] \\
B^{(6)}_{t,u,v} \arrow[d, dash] & & B^{(6)}_{\delta,u,v} \arrow[ll, tail] \\
B^{(5)}_{t,u} \arrow[d, dash] & & \\
B^{(4)}_{0,t} \arrow[rr, tail] & & B^{(4)}_{s,t} \arrow[d, dash] \\
 & & B^{(3)}_{s} \arrow[d, dash] \\
A^{(2)}_{r} \arrow[dr, dash] & & B^{(2)} \arrow[dl, dash] \\
 & A^{(1)} &
\end{tikzcd}
$$

where $\delta = 0$ or 1

This is the homomorphism diagram for uniserial $\mathbb{F}\mathrm{Mod}_{16}$ modules:

$$
\begin{tikzcd}
\begin{array}{c} B^{(8)}_{u+\lambda,v+\lambda+\lambda^2,w+\nu+\lambda^2,x+\nu+\lambda^4} \\ \lambda,\nu\in\mathbb{F} \end{array} \arrow[d, dash] & & \\
\begin{array}{c} B^{(7)}_{1,u+1+\lambda,v+\lambda+\lambda^2,\nu} \\ \lambda,\nu\in\mathbb{F} \end{array} \arrow[rr, tail] & & \begin{array}{c} B^{(7)}_{\delta,u+1+\lambda,v+\lambda+\lambda^2,\nu} \\ \lambda,\nu\in\mathbb{F} \\ \delta\in\{0,1\} \end{array} \arrow[d, dash] \\
\begin{array}{c} B^{(6)}_{t,u+\lambda,v+\lambda+\lambda^2} \\ \lambda\in\mathbb{F} \end{array} \arrow[d, dash] & & \begin{array}{c} B^{(6)}_{\delta+1,u+\lambda,v+\lambda+\lambda^2} \\ \lambda\in\mathbb{F} \\ \delta\in\{0,1\} \end{array} \arrow[ll, tail] \\
\begin{array}{c} B^{(5)}_{t+1,\mu} \\ \mu\in\mathbb{F} \end{array} \arrow[d, dash] & & \\
B^{(4)}_{0,t} \arrow[rr, tail] & & B^{(4)}_{s,t} \arrow[d, dash] \\
 & & B^{(3)}_{s} \arrow[d, dash] \\
A^{(2)}_{r} \arrow[dr, dash] & & B^{(2)} \arrow[dl, dash] \\
 & A^{(1)} &
\end{tikzcd}
$$

A basis for $\mathbb{F}\mathrm{Mod}_{16}$ is

$$\begin{array}{ccc}
 & 1 & \\
\bar{x} & & \bar{y} \\
\bar{x}^2 & & \bar{x}\bar{y} \\
\bar{x}^3 & & \bar{x}^2\bar{y} \\
\bar{x}^4 & & \bar{x}^3\bar{y} \\
\bar{x}^5 & & \bar{x}^4\bar{y} \\
\bar{x}^6 & & \bar{x}^5\bar{y} \\
\bar{x}^7 & & \bar{x}^6\bar{y} \\
 & \bar{x}^7\bar{y} &
\end{array}$$

$$A = \begin{array}{ccc}
 & a & \\
b & & c \\
d & & e \\
f & & g \\
h & & i \\
j & & k \\
l & & m \\
n & & p \\
 & q &
\end{array}, \qquad
\bar{x}A = \begin{array}{ccc}
 & \cdot & \\
a & & \cdot \\
b & & c \\
d & & e \\
f & & g \\
h & & i \\
j & & k \\
l & & m \\
 & p &
\end{array}, \qquad
\bar{y}A = \begin{array}{ccc}
 & \cdot & \\
\cdot & & a \\
\cdot & & b \\
\cdot & & d \\
b & & f \\
b & & b+e+h \\
f & & b+e+j \\
f & & f+i+l \\
 & f+i+n &
\end{array}$$

$$A\bar{x} = \begin{array}{ccc}
 & \cdot & \\
a & & \cdot \\
b & & c \\
d & & e \\
c+f & & g \\
c+e+h & & c+i \\
e+g+j & & c+e+k \\
g+i+l & & e+g+m \\
 & g+i+p &
\end{array}, \qquad
A\bar{y} = \begin{array}{ccc}
 & \cdot & \\
\cdot & & a \\
\cdot & & b \\
\cdot & & d \\
\cdot & & f \\
\cdot & & h \\
\cdot & & j \\
\cdot & & l \\
 & n &
\end{array}$$

The uniserial module of length 1:

$$A^{(1)} = \begin{array}{ccc}
 & \cdot & \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
 & 1 &
\end{array}, \qquad
\bar{x}A^{(1)} = \bar{y}A^{(1)} = A^{(1)}\bar{x} = A^{(1)}\bar{y} = \begin{array}{ccc}
 & \cdot & \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
 & \cdot &
\end{array}$$

The uniserial modules of length 2:

$$A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & r \\ & \cdot & \end{array}, \qquad \bar{x}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & r & \end{array}, \qquad \bar{y}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$A_r^{(2)}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & r & \end{array}, \qquad A_r^{(2)}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad \bar{x}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}, \qquad \bar{y}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

$$B^{(2)}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}, \qquad B^{(2)}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

The uniserial modules of length 3:

$$B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad \bar{y}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & s & \end{array}$$

$$B_s^{(3)}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad B_s^{(3)}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & s & \end{array}$$

The uniserial modules of length 4:

$$B_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ t & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & s \\ & t & \end{array}$$

$$B_{s,t}^{(4)}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad B_{s,t}^{(4)}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & s \\ & t & \end{array}$$

The uniserial modules of length 5:

$$B^{(5)}_{t,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ t & & \cdot \\ u & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(5)}_{t,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ t & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(5)}_{t,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1+t \\ & 1+u & \end{array}$$

$$B^{(5)}_{t,u}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ t+1 & & \cdot \\ & 1 & \end{array}, \qquad B^{(5)}_{t,u}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & t \\ & u & \end{array}$$

The uniserial modules of length 6:

$$B^{(6)}_{t,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ t & & \cdot \\ u & & \cdot \\ v & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(6)}_{t,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ t & & \cdot \\ u & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(6)}_{t,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & t \\ \cdot & & u \\ & v & \end{array}$$

$$B^{(6)}_{t,u,v}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ t+1 & & \cdot \\ u+1 & & 1 \\ & 1 & \end{array}, \qquad B^{(6)}_{t,u,v}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & t \\ \cdot & & u \\ & v & \end{array}$$

The uniserial modules of length 7:

Note: $\delta = 0$ or 1.

$$B^{(7)}_{\delta,u,v,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ \delta & & \cdot \\ u & & \cdot \\ v & & \cdot \\ w & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(7)}_{\delta,u,v,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ \delta & & \cdot \\ u & & \cdot \\ v & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(7)}_{\delta,u,v,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1+\delta \\ \cdot & & 1+u \\ \cdot & & v \\ & w & \end{array}$$

$$B^{(7)}_{\delta,u,v,w}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ \delta+1 & & \cdot \\ u+1 & & 1 \\ v & & 1 \\ & \cdot & \end{array}, \qquad B^{(7)}_{\delta,u,v,w}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \delta \\ \cdot & & u \\ \cdot & & v \\ & w & \end{array}$$

The uniserial modules of length 8:

$$B^{(8)}_{u,v,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ v & & \cdot \\ w & & \cdot \\ x & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(8)}_{u,v,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ v & & \cdot \\ w & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(8)}_{u,v,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & v \\ \cdot & & w \\ & x & \end{array}$$

$$B^{(8)}_{u,v,w,x}\bar{x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & 1 \\ \cdot & & \cdot \\ 1 & & \cdot \\ u+1 & & 1 \\ v & & 1 \\ w & & \cdot \\ & \cdot & \end{array}, \qquad B^{(8)}_{u,v,w,x}\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & v \\ \cdot & & w \\ & x & \end{array}$$

There are no uniserial modules of length 9.

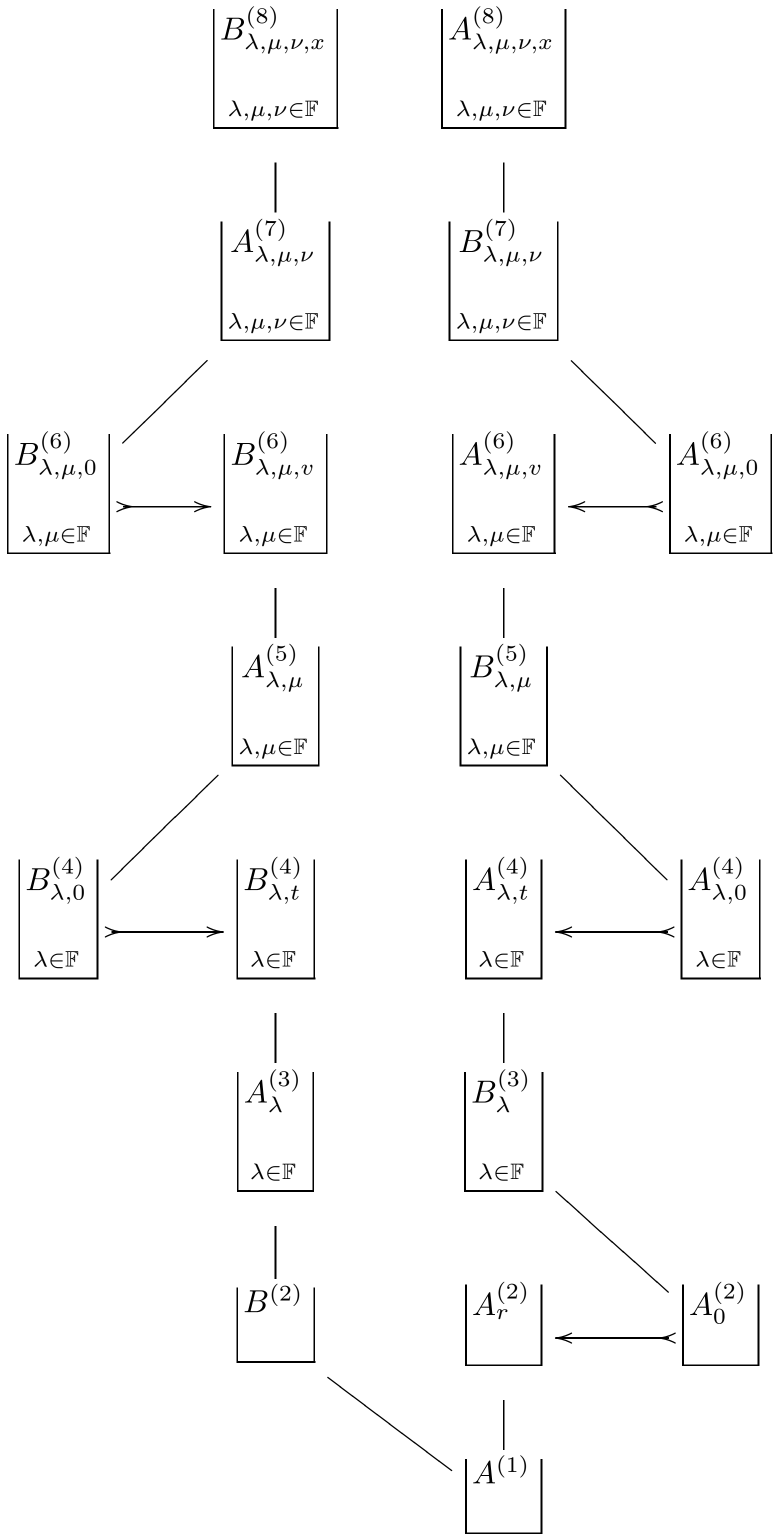
§4 Uniserial submodules of $\mathbb{F}\mathbf{D}_{16}$

Group: $D_{16} = \langle x, y \mid x^2 = y^2 = (xy)^8 = 1 \rangle$

This is the partially ordered set of uniserial $\mathbb{F}D_{16}$ submodules:

$$
\begin{tikzcd}
A^{(8)}_{s,u,w,x} \arrow[d, no head] & & & B^{(8)}_{s,u,w,x} \arrow[d, no head] \\
A^{(7)}_{s,u,w} \arrow[d, no head] & & & B^{(7)}_{s,u,w} \arrow[d, no head] \\
A^{(6)}_{s,u,0} \arrow[r, tail] & A^{(6)}_{s,u,v} \arrow[d, no head] & B^{(6)}_{s,u,v} \arrow[d, no head] & B^{(6)}_{s,u,0} \arrow[l, tail] \\
 & A^{(5)}_{s,u} \arrow[d, no head] & B^{(5)}_{s,u} \arrow[d, no head] & \\
A^{(4)}_{s,t} \arrow[d, no head] & A^{(4)}_{s,0} \arrow[l, tail] & B^{(4)}_{s,0} \arrow[r, tail] & B^{(4)}_{s,t} \arrow[d, no head] \\
A^{(3)}_{s} \arrow[d, no head] & & & B^{(3)}_{s} \arrow[d, no head] \\
A^{(2)}_{0} \arrow[r, tail] & A^{(2)}_{r} \arrow[dr, no head] & & B^{(2)} \arrow[dl, no head] \\
 & & A^{(1)} &
\end{tikzcd}
$$

This is the homomorphism diagram for uniserial $\mathbb{F}D_{16}$ modules:



For this group ring we use a basis which is not in Jennings' form. The group $G = D_{2n} = \left\langle x, y \;\middle|\; x^2 = y^2 = (xy)^n = 1 \right\rangle$ is the dihedral group of order $2n$. We write $\bar{x} = x - 1$ and $\bar{y} = y - 1$. Let n be a power of 2, and $\mathbb{F}$ a field of characteristic 2. Let $a_m = \bar{x}\bar{y}\bar{x}\cdots$ be an element of $\mathbb{F}D_{2n}$, where $\bar{x}$ and $\bar{y}$ appear alternately in the product, starting with $\bar{x}$, and the total number of terms in this product is m. Let $b_m = \bar{y}\bar{x}\bar{y}\cdots$ be defined in a like manner, where the product starts with $\bar{y}$. We define $a_0 = b_0 = 1$. It may be shown that

i) $a_n = b_n = \|D_{2n}\|$

ii) $a_m = b_m = 0$ when $m > n$

iii) a_m and b_m are elements of $\mathrm{Rad}^m(\mathbb{F}D_{2n}$

iv) a_m and b_m are not elements of $\mathrm{Rad}^{m+1}(\mathbb{F}D_{2n}$ if $m \le n$

v) The set $\{a_0, a_i, b_i, a_n \mid 0 < i < n\}$ forms a basis for $\mathbb{F}D_{2n}$ filtered with respect to the powers of the radical, where the subscript indicates the appropriate power of the radical.

We choose not to prove these statements here.

The following results may be quickly generalized to other dihedral 2-groups.

A basis for $\mathbb{F}D_{16}$ is

$$\begin{array}{ccc}
 & 1 & \\
\bar{x} & & \bar{y} \\
\bar{y}\bar{x} & & \bar{x}\bar{y} \\
\bar{x}\bar{y}\bar{x} & & \bar{y}\bar{x}\bar{y} \\
\bar{y}\bar{x}\bar{y}\bar{x} & & \bar{x}\bar{y}\bar{x}\bar{y} \\
\bar{x}\bar{y}\bar{x}\bar{y}\bar{x} & & \bar{y}\bar{x}\bar{y}\bar{x}\bar{y} \\
\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x} & & \bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y} \\
\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x} & & \bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y} \\
 & \bar{y}\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y} &
\end{array}$$

$$A = \begin{array}{ccc}
 & a & \\
b & & c \\
d & & e \\
f & & g \\
h & & i \\
j & & k \\
l & & m \\
n & & p \\
 & q &
\end{array}
\qquad
\bar{x}A = \begin{array}{ccc}
 & \cdot & \\
a & & \cdot \\
\cdot & & c \\
d & & \cdot \\
\cdot & & g \\
h & & \cdot \\
\cdot & & k \\
l & & \cdot \\
 & p &
\end{array}
\qquad
\bar{y}A = \begin{array}{ccc}
 & \cdot & \\
\cdot & & a \\
b & & \cdot \\
\cdot & & e \\
f & & \cdot \\
\cdot & & i \\
j & & \cdot \\
\cdot & & m \\
 & n &
\end{array}$$

$$A\bar{x} = \begin{array}{ccc} & \cdot & \\ a & & \cdot \\ c & & \cdot \\ e & & \cdot \\ g & & \cdot \\ i & & \cdot \\ k & & \cdot \\ m & & \cdot \\ & p & \end{array} \qquad A\bar{y} = \begin{array}{ccc} & \cdot & \\ \cdot & & a \\ \cdot & & b \\ \cdot & & d \\ \cdot & & f \\ \cdot & & h \\ \cdot & & j \\ \cdot & & l \\ & n & \end{array}$$

The Uniserial module of length 1:

$$A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array} \qquad \bar{x}A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

The Uniserial modules of length 2:

$$A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & r \\ & \cdot & \end{array} \qquad \bar{x}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & r & \end{array} \qquad \bar{y}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array} \qquad \bar{x}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array} \qquad \bar{y}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

The Uniserial modules of length 3:

$$A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ & \cdot & \end{array} \qquad \bar{x}A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ & s & \end{array} \qquad \bar{y}A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

$$B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array} \qquad \bar{x}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & s & \end{array}$$

The Uniserial modules of length 4:

$$A_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & t \\ & \cdot & \end{array} \qquad \bar{x}A_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & t & \end{array} \qquad \bar{y}A_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ & \cdot & \end{array}$$

$$B_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ t & & \cdot \\ & \cdot & \end{array} \qquad \bar{x}B_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}B_{s,t}^{(4)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & t & \end{array}$$

The Uniserial modules of length 5:

$$A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ & \cdot & \end{array} \qquad \bar{x}A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ & u & \end{array} \qquad \bar{y}A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

$$B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ & \cdot & \end{array} \qquad \bar{x}B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ & u & \end{array}$$

The Uniserial modules of length 6:

$$A^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & v \\ & \cdot & \end{array} \qquad \bar{x}A^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & v & \end{array} \qquad \bar{y}A^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ & \cdot & \end{array}$$

$$B^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ v & & \cdot \\ & \cdot & \end{array} \qquad \bar{x}B^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}B^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & v & \end{array}$$

The Uniserial modules of length 7:

$$A^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & \cdot \\ \cdot & & w \\ & \cdot & \end{array} \qquad \bar{x}A^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & \cdot \\ & w & \end{array} \qquad \bar{y}A^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

$$B^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ \cdot & & \cdot \\ w & & \cdot \\ & \cdot & \end{array} \qquad \bar{x}B^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}B^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ \cdot & & \cdot \\ & w & \end{array}$$

The Uniserial modules of length 8:

$$A^{(8)}_{s,u,w,x} = \begin{array}{ccc} & \cdot & \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & \cdot \\ \cdot & & w \\ \cdot & & x \\ & \cdot & \end{array} \qquad \bar{x}A^{(8)}_{s,u,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & x & \end{array} \qquad \bar{y}A^{(8)}_{s,u,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ 1 & & \cdot \\ \cdot & & s \\ \cdot & & \cdot \\ \cdot & & u \\ \cdot & & \cdot \\ \cdot & & w \\ & \cdot & \end{array}$$

$$B^{(8)}_{s,u,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ \cdot & & \cdot \\ w & & \cdot \\ x & & \cdot \\ & \cdot & \end{array} \qquad \bar{x}B^{(8)}_{s,u,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ \cdot & & \cdot \\ u & & \cdot \\ \cdot & & \cdot \\ w & & \cdot \\ & \cdot & \end{array} \qquad \bar{y}B^{(8)}_{s,u,w,x} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & x & \end{array}$$

There is no uniserial module of length 9.

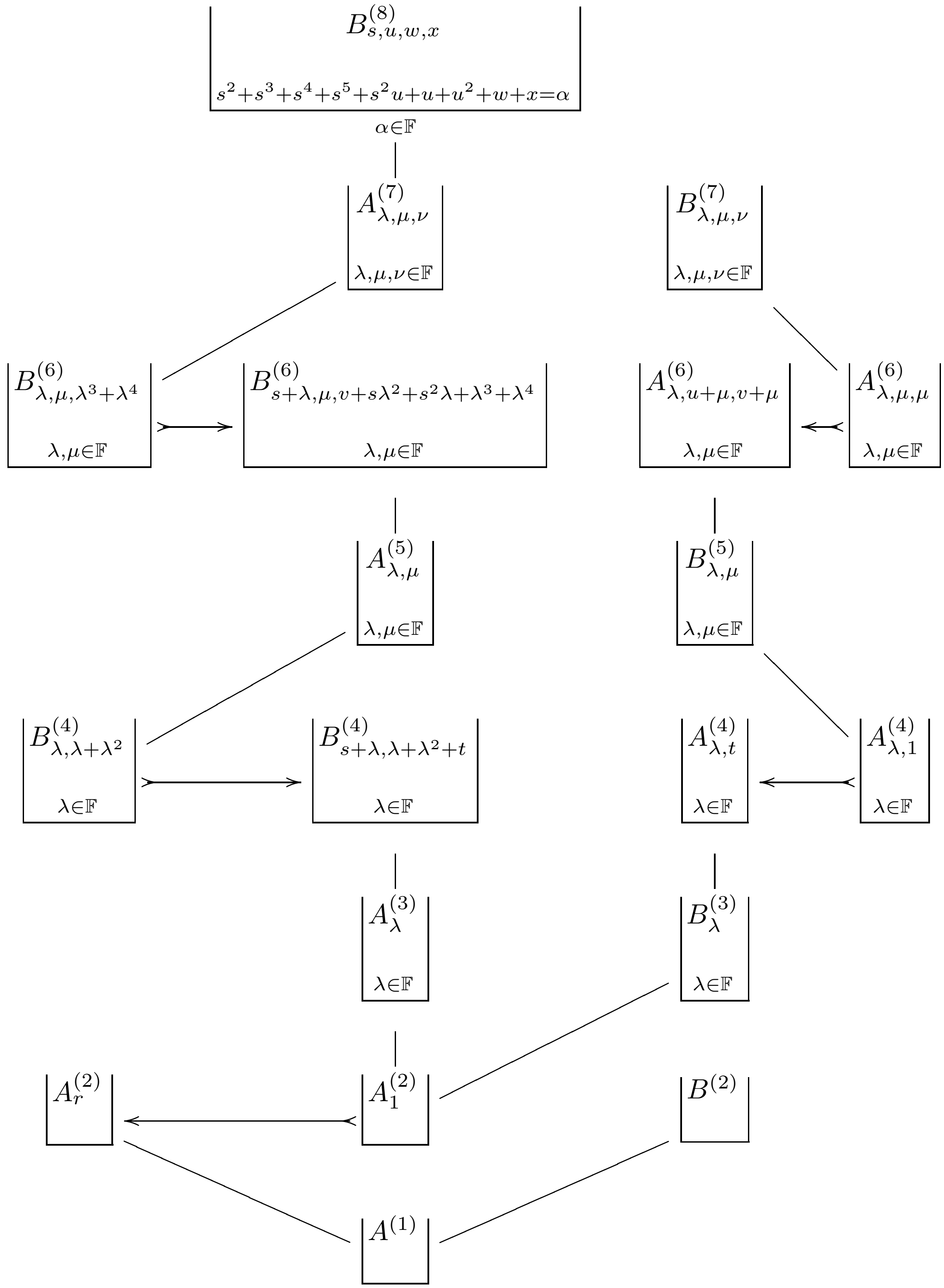
§5 Uniserial submodules of $\mathbb{F}\mathbf{SD}_{16}$

Group: $SD_{16} = \langle x, y \mid x^8 = y^2 = 1,\ {}^{y}x = x^3 \rangle$

This is the partially ordered set of uniserial $\mathbb{F}SD_{16}$ submodules:

$$
\begin{tikzcd}
B^{(8)}_{s,u,w,x} \arrow[d, dash] & & & \\
B^{(7)}_{s,u,w} \arrow[d, dash] & & & A^{(7)}_{s,u,w} \arrow[d, dash] \\
B^{(6)}_{s,u,s^4+s^3} \arrow[r, tail] & B^{(6)}_{s,u,v} \arrow[d, dash] & A^{(6)}_{s,u,v} \arrow[d, dash] & A^{(6)}_{s,u,u} \arrow[l, tail] \\
 & B^{(5)}_{s,u} \arrow[d, dash] & A^{(5)}_{s,u} \arrow[d, dash] & \\
B^{(4)}_{s,t} \arrow[d, dash] & B^{(4)}_{s,s^2+s} \arrow[l, tail] & A^{(4)}_{s,1} \arrow[r, tail] & A^{(4)}_{s,t} \arrow[d, dash] \\
B^{(3)}_{s} \arrow[d, dash] & & & A^{(3)}_{s} \arrow[d, dash] \\
B^{(2)} \arrow[dr, dash] & & A^{(2)}_{r} \arrow[dl, dash] & A^{(2)}_{1} \arrow[l, tail] \\
 & A^{(1)} & &
\end{tikzcd}
$$

This is the homomorphism diagram for uniserial $\mathbb{F}SD_{16}$ modules:



A basis for $\mathbb{F}SD_{16}$ is

$$\begin{array}{cc} \multicolumn{2}{c}{1} \\ \bar{x} & \bar{y} \\ \bar{x}^2 & \bar{x}\bar{y} \\ \bar{x}^3 & \bar{x}^2\bar{y} \\ \bar{x}^4 & \bar{x}^3\bar{y} \\ \bar{x}^5 & \bar{x}^4\bar{y} \\ \bar{x}^6 & \bar{x}^5\bar{y} \\ \bar{x}^7 & \bar{x}^6\bar{y} \\ \multicolumn{2}{c}{\bar{x}^7\bar{y}} \end{array}$$

$$A = \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \\ d & e \\ f & g \\ h & i \\ j & k \\ l & m \\ n & p \\ \multicolumn{2}{c}{q} \end{array}, \qquad \bar{x}A = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ a & \cdot \\ b & c \\ d & e \\ f & g \\ h & i \\ j & k \\ l & m \\ \multicolumn{2}{c}{p} \end{array},$$

$$\bar{y}A = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & a \\ b & b \\ b & b+d+e \\ d+f & b+e+f \\ \cdot & d+f+g+h+i \\ d+f+j & j \\ j & d+f+g+i+j+l+m \\ \multicolumn{2}{c}{j+m+n} \end{array}$$

The Uniserial modules of length 1:

$$A^{(1)} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \multicolumn{2}{c}{1} \end{array}, \qquad \bar{x}A^{(1)} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array}, \qquad \bar{y}A^{(1)} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array}$$

The Uniserial modules of length 2:

$$A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & r \\ & \cdot & \end{array}, \qquad \bar{x}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & r & \end{array}, \qquad \bar{y}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad \bar{x}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}, \qquad \bar{y}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

The Uniserial modules of length 3:

$$A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ & \cdot & \end{array}, \qquad \bar{x}A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ & s & \end{array}, \qquad \bar{y}A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad \bar{y}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & 1+s & \end{array}$$

The Uniserial modules of length 4:

$$A^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & t \\ & \cdot & \end{array}, \qquad \bar{x}A^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ & t & \end{array}, \qquad \bar{y}A^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ 1 & & 1+s \\ & 1+s & \end{array}$$

$$B^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ t & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & s \\ & t & \end{array}$$

The Uniserial modules of length 5:

$$A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ \cdot & & u \\ & \cdot & \end{array}, \qquad \bar{x}A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ & u & \end{array}, \qquad \bar{y}A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s & & \cdot \\ u & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & s \\ s & & 1+s^2 \\ & s+u & \end{array}$$

The Uniserial modules of length 6:

$$A^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ \cdot & & u \\ \cdot & & v \\ & \cdot & \end{array}, \qquad \bar{x}A^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ \cdot & & u \\ & v & \end{array}, \qquad \bar{y}A^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ 1 & & \cdot \\ \cdot & & s+u \\ & u & \end{array}$$

$$B^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s & & \cdot \\ u & & \cdot \\ v & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s & & \cdot \\ u & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(6)}_{s,u,v} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1+s \\ s^2+s & & s^2+s \\ s^2+s & & 1+s^2+s+u \\ & s^2+s+v & \end{array}$$

The Uniserial modules of length 7:

$$A^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ \cdot & & u \\ \cdot & & u \\ \cdot & & w \\ & \cdot & \end{array}, \qquad \bar{x}A^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ \cdot & & u \\ \cdot & & u \\ & w & \end{array}, \qquad \bar{y}A^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ 1 & & \cdot \\ \cdot & & s+u \\ & u & \end{array}$$

$$B^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s & & \cdot \\ u & & \cdot \\ s^4+s^3 & & \cdot \\ w & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s & & \cdot \\ u & & \cdot \\ s^4+s^3 & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}B^{(7)}_{s,u,w} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & 1+s \\ \cdot & & s^2 \\ s+u & & u \\ u & & s+u+s^4+s^3 \\ & u+w & \end{array}$$

The Uniserial modules of length 8:

$$
B^{(8)}_{s,u,w,x} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & 1 \\ s & \cdot \\ s^2+s & \cdot \\ u & \cdot \\ s^4+s^3 & \cdot \\ w & \cdot \\ x & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array}, \qquad \bar{x}B^{(8)}_{s,u,w,x} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & 1 \\ s & \cdot \\ s^2+s & \cdot \\ u & \cdot \\ s^4+s^3 & \cdot \\ w & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array},
$$

$$
\bar{y}B^{(8)}_{s,u,w,x} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & s \\ s^2 & s^2+s \\ \cdot & s^2+u \\ s^2+s^4+s^3 & s^4+s^3 \\ s^4+s^3 & s^2+s^4+s^3+w \\ \multicolumn{2}{c}{s^4+s^3+x} \end{array}
$$

There is no uniserial module of length 9.

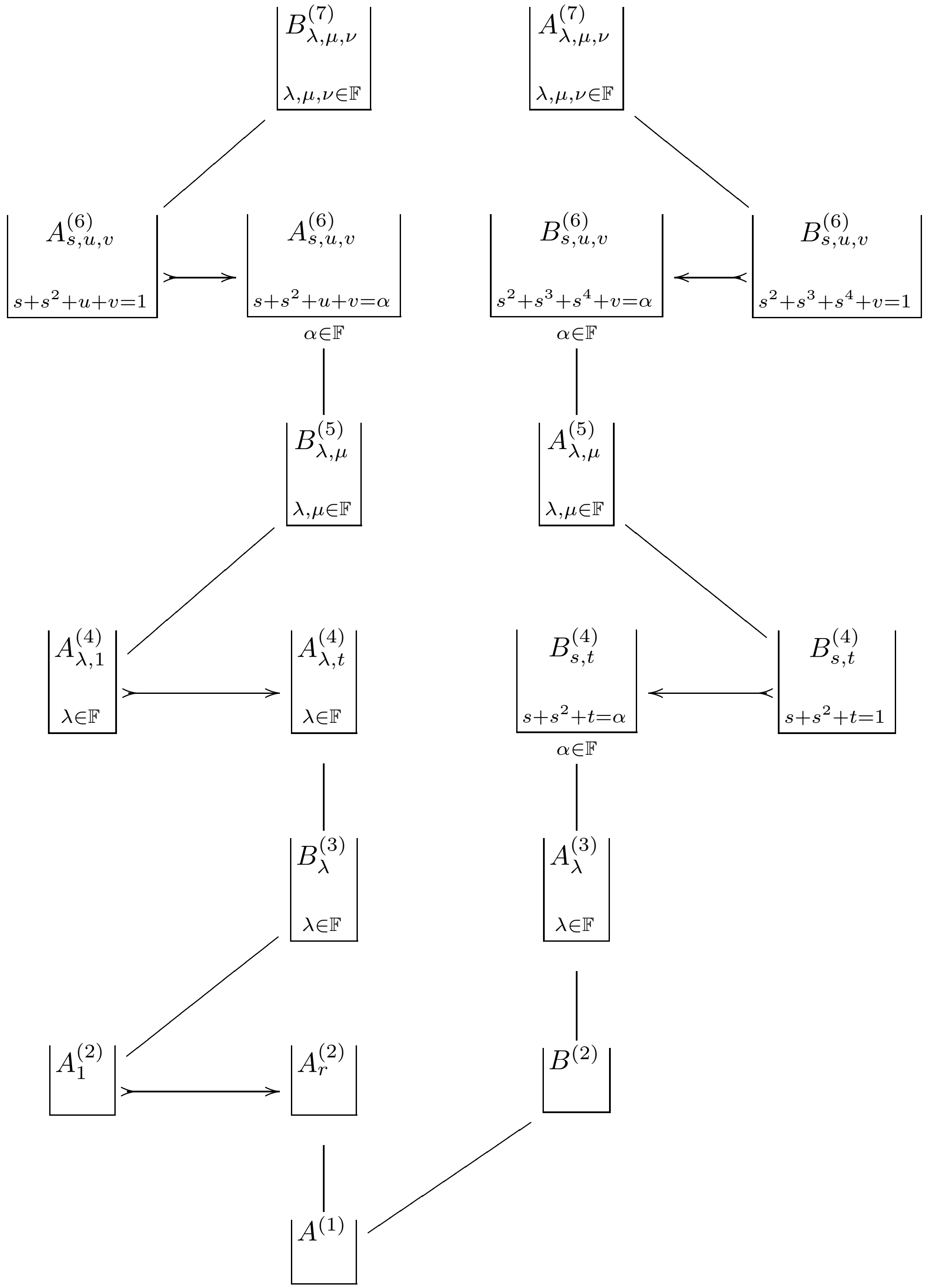
§6 Uniserial submodules of $\mathbb{F}\mathbf{Q}_{16}$

Group: $Q_{16} = \langle x, y \mid x^8 = 1, x^4 = y^2,\ {}^{y}x = x^{-1} \rangle$

This is the partially ordered set of uniserial $\mathbb{F}Q_{16}$ submodules:

$$
\begin{tikzcd}
A^{(7)}_{s,u,w} \arrow[d, dash] & & & B^{(7)}_{s,u,w} \arrow[d, dash] \\
A^{(6)}_{s,u,s^2+s+u+1} \arrow[r, tail] & A^{(6)}_{s,u,v} \arrow[d, dash] & B^{(6)}_{s,u,v} \arrow[d, dash] & B^{(6)}_{s,u,s^4+s^3+s^2+1} \arrow[l, tail] \\
A^{(5)}_{s,t,u} \arrow[d, dash] & A^{(5)}_{s,1,u} \arrow[l, tail] & B^{(5)}_{s,u} \arrow[d, dash] & \\
A^{(4)}_{s,t} \arrow[d, dash] & & B^{(4)}_{s,s^2+s+1} \arrow[r, tail] & B^{(4)}_{s,t} \arrow[d, dash] \\
A^{(3)}_{s} \arrow[d, dash] & & & B^{(3)}_{s} \arrow[d, dash] \\
A^{(2)}_{1} \arrow[r, tail] & A^{(2)}_{r} \arrow[dr, dash] & & B^{(2)} \arrow[dl, dash] \\
& & A^{(1)} &
\end{tikzcd}
$$

This is the homomorphism diagram for uniserial $\mathbb{F}Q_{16}$ modules:



A basis for $\mathbb{F}Q_{16}$ is

$$\begin{array}{ccc} & 1 & \\ \bar{x} & & \bar{y} \\ \bar{x}^2 & & \bar{x}\bar{y} \\ \bar{x}^3 & & \bar{x}^2\bar{y} \\ \bar{x}^4 & & \bar{x}^3\bar{y} \\ \bar{x}^5 & & \bar{x}^4\bar{y} \\ \bar{x}^6 & & \bar{x}^5\bar{y} \\ \bar{x}^7 & & \bar{x}^6\bar{y} \\ & \bar{x}^7\bar{y} & \end{array}$$

$$A = \begin{array}{ccc} & a & \\ b & & c \\ d & & e \\ f & & g \\ h & & i \\ j & & k \\ l & & m \\ n & & p \\ & q & \end{array}, \qquad \bar{x}A = \begin{array}{ccc} & \cdot & \\ a & & \cdot \\ b & & c \\ d & & e \\ f & & g \\ h & & i \\ j & & k \\ l & & m \\ & p & \end{array},$$

$$\bar{y}A = \begin{array}{ccc} & \cdot & \\ \cdot & & a \\ b & & b \\ b & & b+d+e \\ b+c+d+f & & b+e+f \\ b+e & & b+d+e+f+g+h+i \\ b+d+e+g+j & & b+e+j \\ b+e+f+i+j & & b+d+e+g+j+l+m \\ & b+e+f+i+j+m+n & \end{array}$$

The Uniserial modules of length 1:

$$A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}, \qquad \bar{x}A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}, \qquad \bar{y}A^{(1)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

The Uniserial modules of length 2:

$$A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & r \\ & \cdot & \end{array}, \qquad \bar{x}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & r & \end{array}, \qquad \bar{y}A_r^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad \bar{x}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}, \qquad \bar{y}B^{(2)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & \cdot & \end{array}$$

The Uniserial modules of length 3:

$$A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ & \cdot & \end{array}, \qquad \bar{x}A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ & s & \end{array}, \qquad \bar{y}A_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ & 1 & \end{array}$$

$$B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad \bar{x}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & \cdot & \end{array}, \qquad \bar{y}B_s^{(3)} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ & 1+s & \end{array}$$

The Uniserial modules of length 4:

$$
A^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & t \\ & \cdot & \end{array}, \qquad
\bar{x}A^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ & t & \end{array}, \qquad
\bar{y}A^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ 1 & & 1+s \\ & 1+s & \end{array}
$$

$$
B^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ t & & \cdot \\ & \cdot & \end{array}, \qquad
\bar{x}B^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ & \cdot & \end{array}, \qquad
\bar{y}B^{(4)}_{s,t} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & s \\ & t & \end{array}
$$

The Uniserial modules of length 5:

$$
A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ \cdot & & u \\ & \cdot & \end{array}, \qquad
\bar{x}A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ \cdot & & s \\ \cdot & & 1 \\ & u & \end{array}, \qquad
\bar{y}A^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & & 1 \\ & \cdot & \end{array}
$$

$$
B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s+1 & & \cdot \\ u & & \cdot \\ & \cdot & \end{array}, \qquad
\bar{x}B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & \cdot \\ s^2+s+1 & & \cdot \\ & \cdot & \end{array}, \qquad
\bar{y}B^{(5)}_{s,u} = \begin{array}{ccc} & \cdot & \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & 1 \\ s & & s \\ 1+s & & s^2+1 \\ & 1+s+u & \end{array}
$$

The Uniserial modules of length 6:

$$A^{(6)}_{s,u,v} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \\ \cdot & s \\ \cdot & 1 \\ \cdot & u \\ \cdot & v \\ \multicolumn{2}{c}{\cdot} \end{array}, \qquad \bar{x}A^{(6)}_{s,u,v} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \\ \cdot & s \\ \cdot & 1 \\ \cdot & u \\ \multicolumn{2}{c}{v} \end{array}, \qquad \bar{y}A^{(6)}_{s,u,v} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \\ \cdot & s \\ 1 & \cdot \\ 1{+}s & 1{+}u \\ \multicolumn{2}{c}{1{+}s{+}u} \end{array}$$

$$B^{(6)}_{s,u,v} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 1 \\ s & \cdot \\ s^2+s+1 & \cdot \\ u & \cdot \\ v & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array}, \qquad \bar{x}B^{(6)}_{s,u,v} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 1 \\ s & \cdot \\ s^2+s+1 & \cdot \\ u & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array}, \qquad \bar{y}B^{(6)}_{s,u,v} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 1+s \\ s^2+s & s^2+s+1 \\ s^2+s+1 & s^2+s+u \\ \multicolumn{2}{c}{s^2+s+1+v} \end{array}$$

The Uniserial modules of length 7:

$$A^{(7)}_{s,u,w} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ 1 & 1 \\ \cdot & s \\ \cdot & 1 \\ \cdot & u \\ \cdot & s^2+s+1+u \\ \cdot & w \\ \multicolumn{2}{c}{\cdot} \end{array},$$

$$\bar{x}A^{(7)}_{s,u,w} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \\ \cdot & s \\ \cdot & 1 \\ \cdot & u \\ \cdot & s^2+s+1+u \\ \multicolumn{2}{c}{w} \end{array}, \qquad \bar{y}A^{(7)}_{s,u,w} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \\ 1 & s+1 \\ s & 1 \\ \cdot & s^2+1+u \\ \multicolumn{2}{c}{s^2+s+1+u} \end{array}$$

$$B^{(7)}_{s,u,w} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & 1 \\ s & \cdot \\ s^2+s+1 & \cdot \\ u & \cdot \\ 1+s^2+s^3+s^4 & \cdot \\ w & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array}, \qquad \bar{x}B^{(7)}_{s,u,w} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 1 \\ s & \cdot \\ s^2+s+1 & \cdot \\ u & \cdot \\ 1+s^2+s^3+s^4 & \cdot \\ \multicolumn{2}{c}{\cdot} \end{array},$$

$$\bar{y}B^{(7)}_{s,u,w} = \begin{array}{cc} \multicolumn{2}{c}{\cdot} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & 1 \\ s & 1+s \\ 1 & s^2 \\ 1+u & 1+u \\ 1+s+u & u+s^2+s^3+s^4 \\ \multicolumn{2}{c}{1+s+u+w} \end{array}$$

There are no uniserial modules of length 8.

§7 Uniserial submodules of $\mathbb{F}\mathbf{D}_8\mathbf{Y4}$

Group: $D_8\mathrm{Y}4 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, x \in Z, (yz)^2 = x^2 \rangle$, where Z is the center of the group.

This is the partially ordered set of uniserial $\mathbb{F}D_8\mathrm{Y}4$ submodules:

$$\begin{tikzcd}[column sep=small]
A^{(2)}_{r,s} \arrow[dr, dash] & B^{(2)}_r \arrow[d, dash] & C^{(2)} \arrow[dl, dash] \\
& A^{(1)} &
\end{tikzcd}$$

This is the homomorphism diagram for uniserial $\mathbb{F}D_8\mathrm{Y}4$ modules:

$$\begin{tikzcd}[column sep=small]
A^{(2)}_{r,s} \arrow[dr, dash] & B^{(2)}_r \arrow[d, dash] & C^{(2)} \arrow[dl, dash] \\
& A^{(1)} &
\end{tikzcd}$$

As there are no uniserial modules of a length greater than 2, all uniserial modules are mutually non-isomorphic.

A basis for $\mathbb{F}D_8\mathrm{Y}4$ is:

$$\begin{array}{cccccc}
 & 1 & & & & \\
\bar{x} & & \bar{y} & & \bar{z} & \\
\bar{x}^2 & & \bar{x}\bar{y} & \bar{x}\bar{z} & & \bar{y}\bar{z} \\
\bar{x}^3 & & \bar{x}^2\bar{y} & \bar{x}^2\bar{z} & & \bar{x}\bar{y}\bar{z} \\
 & \bar{x}^3\bar{y} & & \bar{x}^3\bar{z} & & \bar{x}^2\bar{y}\bar{z} \\
 & & & & \bar{x}^3\bar{y}\bar{z} &
\end{array}$$

$$A = \begin{array}{cccccc}
 & a & & & & \\
b & & c & & d & \\
e & & f & g & & h \\
i & & j & k & & l \\
 & m & & n & & p \\
 & & & & q &
\end{array}, \qquad
\bar{x}A = \begin{array}{cccccc}
 & \cdot & & & & \\
a & & \cdot & & \cdot & \\
b & & c & d & & \cdot \\
e & & f & g & & h \\
 & j & & k & & l \\
 & & & & p &
\end{array},$$

$$\bar{y}A = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & a & & \cdot & \\ \cdot & & b & \cdot & & d \\ \cdot & & e & \cdot & & g \\ & i & & \cdot & & k \\ & & & & n & \end{array}$$

$$\bar{z}A = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & a & \\ c & & \cdot & b & & c \\ f & & c & c+e+h & & f \\ & f & & f+i+l & & c+h+j \\ & & & & f+l+m & \end{array}$$

The Uniserial module of length 1:

$$A^{(1)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & 1 & \end{array}, \qquad \bar{x}A^{(1)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \end{array},$$

$$\bar{y}A^{(1)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \end{array} \qquad \bar{z}A^{(1)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \end{array}$$

The Uniserial modules of length 2:

$$A^{(2)}_{r,s} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & r & & s & & 1 \\ & & & & \cdot & \end{array}, \qquad \bar{x}A^{(2)}_{r,s} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & 1 & \end{array},$$

$$\bar{y}A^{(2)}_{r,s} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & s & \end{array} \qquad \bar{z}A^{(2)}_{r,s} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & r & \end{array}$$

$$B_r^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & 1 & & r & & \cdot \\ & & & & \cdot & \end{array}, \qquad \bar{x}B_r^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \end{array},$$

$$\bar{y}B_r^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & r & \end{array} \qquad \bar{z}B_r^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & 1 & \end{array}$$

$$C^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & 1 & & \cdot \\ & & & & \cdot & \end{array}, \qquad \bar{x}C^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \end{array},$$

$$\bar{y}C^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & 1 & \end{array}, \qquad \bar{z}C^{(2)} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \end{array}$$

There are no uniserial modules of length 3.

Chapter 3: Tensor decompositions of the Regular Representation

In this chapter we present several theorems related to tensor decompositions of group rings, culminating in a description of tensor decompositions of the group rings of particular groups of order 16. Throughout, $\mathbb{F}$ is a field of characteristic p, where p is a prime. Unless otherwise noted, G is a p-group, and H and K are subgroups of G. We define M and N to be left $\mathbb{F}G$-modules, and we denote the Loewy length of the module M by $\ell\ell(M)$.

The tensor product $M \otimes N$ is the set of $\mathbb{F}$-linear combinations of the elements $m \otimes_{\mathbb{F}} n$, where $m \in M$ and $n \in N$. The group G acts on $M \otimes N$ by $g \cdot m \otimes n = gm \otimes gn$, for any element g of G. Extending $\mathbb{F}$-linearly gives $M \otimes N$ a left $\mathbb{F}G$-module structure. To distinguish multiplication of $m \otimes n$ by g from multiplication of m by g, we will write $g \cdot m \otimes n$ and gm respectively.

Two of the more important results of this section deal with the particular situation where $\mathbb{F}G = M \otimes N$, and $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$. Under these circumstances, Theorem 1.6 states that the radical and socle series of M coincide, as do those of N. In Theorem 1.7 we see that the quotient of consecutive powers of the radical of $\mathbb{F}G$ may be written entirely in terms of M and N.

We note here that Theorems 1.5 through 1.7, and Corollary 1.8 have immediate generalizations, without significantly changing the proofs. We may replace the supposition that $\mathbb{F}G \cong M \otimes N$ with the supposition that $M \otimes N$ is a module for which the radical series and the socle series coincide. The conjecture after Corollary 1.8 may likewise be broadened in a number of ways.

§1 Tensor Decompositions

In the following two lemmas we do not require G to be a p-group.

Lemma 1.1. *If g is an element of the group G, and m and n are elements of left $\mathbb{F}G$-modules, then*

$$(g-1)\cdot m\otimes n = (g-1)m\otimes n + m\otimes (g-1)n + (g-1)m\otimes (g-1)n.$$

Proof. Care must be taken in noting the differences between the two definitions of multiplication.

$$\begin{aligned}(g-1)\cdot m\otimes n &= g\cdot m\otimes n - m\otimes n = gm\otimes gn - m\otimes n\\ &= gm\otimes gn - m\otimes n \quad - gm\otimes n + gm\otimes n\\ &= gm\otimes (g-1)n + (g-1)m\otimes n - m\otimes (g-1)n + m\otimes (g-1)n\\ &= (g-1)m\otimes (g-1)n + (g-1)m\otimes n + m\otimes (g-1)n\end{aligned}$$

□

Lemma 1.2. *The product of the augmentation ideal and $M\otimes N$, $IG\cdot M\otimes N$, is contained in $IG\ M\otimes N + M\otimes IG\ N$.*

Proof. An element of $M\otimes N$ is a linear combination of elements of the form $m\otimes n$, where m and n are elements of some fixed bases for M and N, respectively. A basis for the augmentation ideal IG is the set $\{g-1 \mid 1\neq g\in G\}$, so $IG\cdot M\otimes N$ consists of linear combinations of elements of the form $(g-1)\cdot m\otimes n$. From the previous lemma, $(g-1)\cdot m\otimes n = (g-1)m\otimes n + m\otimes (g-1)n + (g-1)m\otimes (g-1)n$, which is the sum of elements of $IG\ M\otimes N$, $M\otimes IG\ N$, and $IG\ M\otimes IG\ N$. Since $IG\ M\otimes IG\ N$ is contained in both $IG\ M\otimes N$ and $M\otimes IG\ N$, the lemma is proved. □

If we again require G to be a p-group, then the radical and the augmentation ideal coincide. The following results are obtained.

Lemma 1.3. *If M and N are left $\mathbb{F}G$-modules, then*

1) $\mathrm{Rad}^k(M \otimes N) \subseteq \sum_{i=0}^{k} \mathrm{Rad}^i(M) \otimes \mathrm{Rad}^{k-i}(N)$

2) $\mathrm{Soc}^k(M \otimes N) \supseteq \sum_{i=0}^{k} \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{k+1-i}(N)$

Proof. For statement 1), when $k = 0$ we have equality. Assume that statement 1) is true for all $k \leq K$. Then

$$\begin{aligned}
\mathrm{Rad}^{K+1}(M \otimes N) &= IG \cdot \mathrm{Rad}^K(M \otimes N) \\
&\subseteq \sum_{i=0}^{K} IG \cdot \mathrm{Rad}^i(M) \otimes \mathrm{Rad}^{K-i}(N) \\
&\subseteq \sum_{i=0}^{K} \mathrm{Rad}^{i+1}(M) \otimes \mathrm{Rad}^{K-i}(N) + \mathrm{Rad}^i(M) \otimes \mathrm{Rad}^{K+1-i}(N) \\
&= \sum_{i=0}^{K+1} \mathrm{Rad}^i(M) \otimes \mathrm{Rad}^{K+1-i}(N).
\end{aligned}$$

As for statement 2), it is evidently true when $k = 0$. Assume that statement 2) is true for all $k < K$. With $k = K$, we work with the radical of any summand of the right side.

$$\begin{aligned}
\mathrm{Rad}(\mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{K+1-i}(N)) &= IG \cdot \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{K+1-i}(N) \\
&\subseteq IG\,\mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{K+1-i}(N) + \mathrm{Soc}^i(M) \otimes IG\,\mathrm{Soc}^{K+1-i}(N) \\
&\subseteq \mathrm{Soc}^{i-1}(M) \otimes \mathrm{Soc}^{K+1-i}(N) + \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{K-i}(N) \\
&\subseteq \mathrm{Soc}^{K-1}(M \otimes N).
\end{aligned}$$

Since the radical of $\mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{K+1-i}(N)$ is contained in $\mathrm{Soc}^{K-1}(M \otimes N)$, we see that $\mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{K+1-i}(N)$ is contained in $\mathrm{Soc}^K(M \otimes N)$. □

Lemma 1.4. *The Loewy length of the tensor product of $\mathbb{F}G$-modules $M \otimes N$ has an upper bound, $\ell\ell(M \otimes N) \leq \ell\ell(M) + \ell\ell(N) - 1$.*

Proof. The Loewy length of a module M is defined as the minimum power k such that $\mathrm{Rad}^k(M) = 0$. Using the previous lemma, any element of $\mathrm{Rad}^k(M \otimes N)$

is a sum of elements in the modules $\mathrm{Rad}^i(M) \otimes \mathrm{Rad}^{k-i}(N)$. To obtain a non-zero element in the latter tensor product we must have $i < \ell\ell(M)$ and $k - i < \ell\ell(N)$. Thus $\ell\ell(M \otimes N) - 1 \leq \ell\ell(M) - 1 + \ell\ell(N) - 1$. □

Theorem 1.5. *Assume $\mathbb{F}G \cong M \otimes N$. Then $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$ if and only if the inclusions of Lemma 1.3 become equalities.*

Proof. Let $\lambda = \ell\ell(\mathbb{F}G)$, $\mu = \ell\ell(M)$, and $\nu = \ell\ell(N)$. By Lemma 1.4, $\lambda \leq \mu + \nu - 1$. As G is a p-group, the radical and socle series of $\mathbb{F}G$ coincide. In other words, $\mathrm{Soc}^i(\mathbb{F}G) = \mathrm{Rad}^{\lambda-i}(\mathbb{F}G)$. Using this, Lemma 1.3 implies

$$\begin{aligned}\sum_{i=0}^{k} \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{k+1-i}(N) \subseteq \mathrm{Soc}^k(\mathbb{F}G) &= \mathrm{Rad}^{\lambda-k}(\mathbb{F}G) \\ &\subseteq \sum_{j=0}^{\lambda-k} \mathrm{Rad}^{\lambda-k-j}(M) \otimes \mathrm{Rad}^j(N),\end{aligned} \tag{$*$}$$

for any k between 0 and λ. As $\mathrm{Rad}^{\mu}(M) = 0$, and $\mathrm{Rad}^{\nu}(N) = 0$, we know that

$$\mathrm{Rad}^{\lambda-k-j}(M) \otimes \mathrm{Rad}^j(N) \subseteq \mathrm{Soc}^{\mu+k+j-\lambda}(M) \otimes \mathrm{Soc}^{\nu-j}(N). \tag{$**$}$$

First, we assume that $\lambda = \mu + \nu - 1$. Inclusion $(**)$ becomes

$$\mathrm{Rad}^{\mu+\nu-1-k-j}(M) \otimes \mathrm{Rad}^j(N) \subseteq \mathrm{Soc}^{j+k+1-\nu}(M) \otimes \mathrm{Soc}^{\nu-j}(N),$$

which, when combined with the inclusions $(*)$ gives

$$\begin{aligned}\sum_{i=0}^{k} \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{k+1-i}(N) &\subseteq \sum_{j=0}^{\mu+\nu-1-k} \mathrm{Rad}^{\mu+\nu-1-k-j}(M) \otimes \mathrm{Rad}^j(N) \\ &\subseteq \sum_{j=0}^{\mu+\nu-1-k} \mathrm{Soc}^{j+k+1-\nu}(M) \otimes \mathrm{Soc}^{\nu-j}(N).\end{aligned}$$

Replacing j with $i + \nu - k - 1$, the last module becomes

$$\sum_{i=k+1-\nu}^{\mu} \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{k+1-i}(N).$$

We know that $\mathrm{Soc}^k(M) = 0$ if $k \leq 0$, and $\mathrm{Soc}^k(M) = M$ if $k \geq \ell\ell(M)$, for any module M. Thus we have

$$\sum_{i=0}^{k} \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{k+1-i}(N) = \sum_{i=k+1-\nu}^{\mu} \mathrm{Soc}^i(M) \otimes \mathrm{Soc}^{k+1-i}(N),$$

which implies that the inclusions of $(*)$ are equalities.

Now we assume that the set inclusions of $(*)$ are equalities. With $k = 0$ we obtain

$$0 = \mathrm{Rad}^{\lambda}(\mathbb{F}G) = \sum_{j=0}^{\lambda} \mathrm{Rad}^{\lambda-j}(M) \otimes \mathrm{Rad}^{j}(N).$$

But if $\lambda - j \leq \mu - 1$ and $j \leq \nu - 1$, the summation on the right can not be zero. Thus $\mu - 1 + \nu - 1 \leq \lambda - 1$. But we know from Lemma 1.4 that $\lambda \leq \mu + \nu - 1$, so we must have equality. $\square$

Remark: We see from the proof of Theorem 1.5 that equality in condition 1) of Lemma 1.3, when $k = \lambda$ by itself is enough to imply that $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$, which in turn implies equality in conditions 1) and 2) of Lemma 1.3. Thus, if $\mathbb{F}G \cong M \otimes N$, and $0 = \mathrm{Rad}^{\lambda}(M \otimes N) = \sum_{j=0}^{\lambda} \mathrm{Rad}^{\lambda-j}(M) \otimes \mathrm{Rad}^{j}(N)$, where $\lambda = \ell\ell(M \otimes N)$, then $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$, and equality holds in the inclusions of Lemma 1.3.

Theorem 1.6. *If $\mathbb{F}G \cong M \otimes N$, and $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$, then the socle series and radical series of M coincide, as do those of N.*

Proof. We are given that $\mathbb{F}G \cong M \otimes N$. Let $\mu = \ell\ell(M)$, $\nu = \ell\ell(N)$, and $\lambda = \ell\ell(\mathbb{F}G) = \mu + \nu - 1$. By Theorem 1.5 we have

$$\begin{aligned} \mathrm{Soc}^1(M) \otimes \mathrm{Soc}^1(N) &= \mathrm{Soc}^1(M \otimes N) \\ &= \mathrm{Rad}^{\lambda-1}(M \otimes N) = \sum_{j=0}^{\lambda-1} \mathrm{Rad}^{\lambda-1-j}(M) \otimes \mathrm{Rad}^{j}(N). \end{aligned}$$

As $\mathrm{Rad}^{\nu}(N) = 0$, any summand with $j \geq \nu$ is zero. Also, as $\mathrm{Rad}^{\mu}(M) = 0$, the non-zero terms in the sum occur when $\mu - 1 \geq \lambda - 1 - j = \mu + \nu - 2 - j$, which gives $j \geq \nu - 1$. Thus the only non-zero term in this sum occurs when $j = \nu - 1$, and we have

$$\mathrm{Soc}^1(M) \otimes \mathrm{Soc}^1(N) = \mathrm{Rad}^{\mu-1}(M) \otimes \mathrm{Rad}^{\nu-1}(N).$$

We must have $\mathrm{Soc}^1(M) = \mathrm{Rad}^{\mu-1}(M)$ and $\mathrm{Soc}^1(N) = \mathrm{Rad}^{\nu-1}(N)$.

Assume that $\mathrm{Soc}^k(M) = \mathrm{Rad}^{\mu-k}(M)$ and $\mathrm{Soc}^k(N) = \mathrm{Rad}^{\nu-k}(N)$, for all $k < K$. We know that

$$\mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N) \subseteq \sum_{j=0}^{\mu+\nu-1-K} \mathrm{Rad}^{\mu+\nu-1-K-j}(M) \otimes \mathrm{Rad}^j(N).$$

Each summand with $j \geq \nu$ is zero. The summand with $j = \nu - 1$ is

$$\mathrm{Rad}^{\mu-K}(M) \otimes \mathrm{Rad}^{\nu-1}(N) \subseteq \mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N). \tag{$\dagger$}$$

The summands with values of j less than $\nu - 1$ are contained in the module

$$\mathrm{Rad}^{\mu-K+1}(M) \otimes \mathrm{Rad}^0(N) = \mathrm{Soc}^{K-1}(M) \otimes N.$$

Given $m \otimes n \in \mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N)$, we may write $m \otimes n = a + b$, where $a \in \mathrm{Rad}^{\mu-K}(M) \otimes \mathrm{Soc}^1(N)$, and $b \in \mathrm{Soc}^{K-1}(M) \otimes N$. As $a \in \mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N)$, $b = m \otimes n - a$ must also be an element of $\mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N)$. Thus, as these are tensor products of vector spaces,

$$\begin{aligned} b \in (\mathrm{Soc}^{K-1}(M) \otimes N) \cap (\mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N)) &= \mathrm{Soc}^{K-1}(M) \otimes \mathrm{Soc}^1(N) \\ &= \mathrm{Rad}^{\mu+1-K}(M) \otimes \mathrm{Rad}^{\nu-1}(N) \subseteq \mathrm{Rad}^{\mu-K}(M) \otimes \mathrm{Rad}^{\nu-1}(N). \end{aligned}$$

Thus $m \otimes n \in \mathrm{Rad}^{\mu-K}(M) \otimes \mathrm{Soc}^1(N)$, for all elements $m \otimes n$ of $\mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N)$, and the inclusion of ($\dagger$) is reversed, so

$$\mathrm{Soc}^K(M) \otimes \mathrm{Soc}^1(N) = \mathrm{Rad}^{\mu-K}(M) \otimes \mathrm{Rad}^{\nu-1}(N).$$

From this we deduce that $\mathrm{Soc}^K(M) = \mathrm{Rad}^{\mu-K}(M)$. By a symmetric argument, $\mathrm{Soc}^K(N) = \mathrm{Rad}^{\mu-K}(N)$ for all K. $\square$

Theorem 1.7. *If $\mathbb{F}G \cong M \otimes N$ and $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$, then*

$$\operatorname{Rad}^k(\mathbb{F}G)/\operatorname{Rad}^{k+1}(\mathbb{F}G) \cong \bigoplus_{i=0}^{k} \big(\operatorname{Rad}^i(M)/\operatorname{Rad}^{i+1}(M)\big) \otimes \big(\operatorname{Rad}^{k-i}(N)/\operatorname{Rad}^{k-i+1}(N)\big).$$

Proof. We start by creating a homomorphism between the modules. We represent elements of $\operatorname{Rad}^i(M)$ and $\operatorname{Rad}^{k-i}(N)$ by m_i and n_{k-i} respectively. Let ϕ be the map from the direct sum on the right to $\operatorname{Rad}^k(M\otimes N)/\operatorname{Rad}^{k+1}(M\otimes N)$ defined on each term of each summand by

$$\phi\Big(\big(m_i + \operatorname{Rad}^{i+1}(M)\big) \otimes \big(n_{k-i} + \operatorname{Rad}^{k-i+1}(N)\big)\Big) = m_i \otimes n_{k-i} + \operatorname{Rad}^{k+1}(M\otimes N).$$

We extended ϕ $\mathbb{F}$-linearly to the entire sum. This map is clearly a well defined $\mathbb{F}G$-homomorphism. Every element of $\operatorname{Rad}^k(M\otimes N)$ is the sum of elements of the form $m_i \otimes n_{k-i}$ by Theorem 1.5, so ϕ is surjective.

We know that the dimensions of the quotients $\operatorname{Rad}^k(\mathbb{F}G)/\operatorname{Rad}^{k+1}(\mathbb{F}G)$ must add up to $\dim(\mathbb{F}G) = |G|$. Let $r_i = \dim\big(\operatorname{Rad}^i(M)/\operatorname{Rad}^{i+1}(M)\big)$, and let $s_j = \dim\big(\operatorname{Rad}^j(N)/\operatorname{Rad}^{j+1}(N)\big)$. Let $L = \ell\ell(\mathbb{F}G)$, $\mu = \ell\ell(M)$, and $\nu = \ell\ell(N)$. We have

$$\sum_{k=0}^{L} \dim\left(\bigoplus_{i=0}^{k} \big(\operatorname{Rad}^i(M)/\operatorname{Rad}^{i+1}(M)\big) \otimes \big(\operatorname{Rad}^{k-i}(N)/\operatorname{Rad}^{k-i+1}(N)\big)\right) =$$
$$\sum_{k=0}^{L}\sum_{i=0}^{k}(r_i s_{k-i}) = \sum_{i=0}^{\mu}\sum_{j=0}^{\nu} r_i s_i = \dim(M)\dim(N) = |G|.$$

The homomorphism ϕ is a surjection of one module to another where both modules have the same dimension. This implies that ϕ is an isomorphism. □

For an $\mathbb{F}G$-module M, the *Poincaré polynomial* associated to the filtration of M by its radical powers is defined as the polynomial

$$P_M(t) = \sum_{i=0}^{\infty} t^i \dim(\operatorname{Rad}^i(M)/\operatorname{Rad}^{i+1}(M).$$

We note that if $P_M(t)$ is a polynomial of degree k, then there is no power of t less than k for which the coefficient is 0. In terms of Poincaré polynomials, Theorem 1.7 may be restated as, Corollary 1.8. *If* $\mathbb{F}G \cong M \otimes N$ *and* $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$, *then* $P_{\mathbb{F}G}(t) = P_M(t)P_N(t)$.

Conjecture. *If* $\mathbb{F}G \cong M \otimes N$, *then* $\ell\ell(\mathbb{F}G) = \ell\ell(M) + \ell\ell(N) - 1$.

This conjecture holds true for all cases where $|G| \leq 16$.

Lemma 1.9. *Let* M *and* N *be left* $\mathbb{F}G$*-modules. The following statements are equivalent:*

1) $M \otimes N \cong \mathbb{F}G$
2) $\dim(M \otimes N) = |G|$ *and* $M \otimes N$ *may be generated by a single element,*
3) $\dim(M \otimes N) = |G|$ *and* $\|G\| \cdot M \otimes N \neq 0$,
4) $\dim(M \otimes N) = |G|$ *and* $\ell\ell(M \otimes N) = \ell\ell(\mathbb{F}G)$.

Proof. We start by showing that 2), 3) and 4) follow immediately from 1). Assume that $M \otimes N \cong \mathbb{F}G$. The dimension of $\mathbb{F}G$ is $|G|$, and as $M \otimes N \cong \mathbb{F}G$, $\dim(M \otimes N) = |G|$. Let ϕ be an isomorphism, $\phi : \mathbb{F}G \to M \otimes N$. Since 1 generates $\mathbb{F}G$ as a left $\mathbb{F}G$-module, the element $\phi(1)$ also generates $M \otimes N$ as a left $\mathbb{F}G$-module. The product $\|G\| \cdot 1 = \|G\| \neq 0$, so likewise $\|G\| \cdot \phi(1) \neq 0$. Isomorphic modules have the same Loewy length.

Assume that $\dim(M \otimes N) = |G|$, and that there is an element a which generates $M \otimes N$. Let ϕ be the left $\mathbb{F}G$-module homomorphism $\phi : \mathbb{F}G \to M \otimes N$ given by $\phi(x) = x \cdot a$. As $\phi(\mathbb{F}G) = M \otimes N$, and $\dim(M \otimes N) = \dim(\mathbb{F}G)$, the homomorphism must be an isomorphism. Thus condition 2) implies condition 1).

Assume that $\dim(M \otimes N) = |G|$, and that $\|G\| \cdot M \otimes N \neq 0$. This implies that there must exist an element $m \otimes n$ of $M \otimes N$ for which $\|G\| \cdot m \otimes n \neq 0$. Let ϕ be the left $\mathbb{F}G$-module homomorphism $\phi : \mathbb{F}G \to M \otimes N$ given by $\phi(x) = x \cdot m \otimes n$. Then $\phi(\|G\|) = \|G\| \cdot m \otimes n \neq 0$. Since G is a p-group, the socle of $\mathbb{F}G$ is generated by the element $\|G\|$. The kernel of ϕ must be 0, as any larger kernel would intersect the socle non-trivially. As the dimensions of $\mathbb{F}G$ and $M \otimes N$ are the same, the homomorphism ϕ must be an isomorphism. Thus condition 3) implies condition 1).

Assume that $\dim(M \otimes N) = |G|$, and that $\ell\ell(M \otimes N) = \ell\ell(\mathbb{F}G)$. Let $l = \ell\ell(M \otimes N)$. We have $\mathrm{Rad}^l(\mathbb{F}G) \cdot M \otimes N = 0$, but $\mathrm{Rad}^{l-1}(\mathbb{F}G) \cdot M \otimes N \neq 0$. Since G is a p-group, the radical and socle series coincide, and $\mathrm{Rad}^{l-1}(\mathbb{F}G) = \mathrm{Soc}(\mathbb{F}G) = \mathbb{F}\|G\|$. Thus condition 4) implies condition 3). $\square$

Note: Conditions 1) and 2) are equivalent even if G is not a p-group.

We now observe certain characteristics of possible factors in a tensor decomposition.

Lemma 1.10. *If $\mathbb{F}G \cong M \otimes N$ then both M and N are cyclic modules.*

Proof. We assume that $\mathbb{F}G \cong M \otimes N$. By Lemma 1.9, there must be an element $m \otimes n$ which generates $M \otimes N$. Since the dimensions of $\mathbb{F}G$ and $M \otimes N$ is $|G|$, the set $\{g \cdot m \otimes n = gm \otimes gn \mid g \in G\}$ forms a basis for $M \otimes N$. This set is contained in the set $\{gm \otimes hn \mid g, h \in G\}$. The modules spanned by $\{gm | g \in G\}$ and $\{hn | h \in G\}$ form submodules of M and N, respectively. The dimensions of modules are related by the inequalities,

$$\begin{aligned}
\dim(\mathbb{F}G) = |G| &= |\{gm \otimes gn \mid g \in G\}| \\
&\leq \dim(\mathrm{Span}(\{gm \otimes hn \mid g, h \in G\})) \\
&= \dim(\mathrm{Span}(\{gm | g \in G\})) \dim(\mathrm{Span}(\{hn | h \in G\})) \\
&\leq \dim(M) \dim(N) = \dim(M \otimes N) = \dim(\mathbb{F}G).
\end{aligned}$$

We must have equality throughout, and M and N must be generated by m and n respectively. $\square$

We recall the following notation from the introduction.

For a p-group G, Jennings [J, Al 1, Sc] describes a decreasing series of subgroups, $\kappa_i(G) = \{g \in G | g \equiv 1 \text{ modulo } \mathrm{Rad}^i(\mathbb{F}G)\}$, which we refer to as the Jennings series of G. This series of subgroups has the following properties:

1) $[\kappa_\lambda, \kappa_\mu] \subseteq \kappa_{\lambda+\mu}$,
2) $g^p \in \kappa_{ip}$ for all $g \in \kappa_i$,
3) $\kappa_\lambda / \kappa_{2\lambda}$ is elementary abelian.

For each $i \geq 1$ choose elements $x_{i,s}$ of G such that the set $\{x_{i,s}\kappa_{i+1} \mid 1 \leq s \leq d_i\}$ forms a basis for κ_i/κ_{i+1}. Let $\overline{x}_{i,s} = x_{i,s} - 1 \in \mathbb{F}G$. There are $|G|$

products of the form $\prod \overline{x}_{i,s}^{\alpha_{i,s}}$, where the factors are listed in lexicographic order, and $0 \leq \alpha_{i,s} \leq p-1$. The weight of such a product is defined to be $\sum i\alpha_{i,s}$. Jennings' theorem states that the set of products of weight w lie in $\mathrm{Rad}^w(\mathbb{F}G)$, and form a basis modulo $\mathrm{Rad}^{w+1}(G)$.

Alperin comments [Al 1] that the order of the factors is irrelevant. After choosing a particular order for these factors, let $\{\beta_{i,t}\}$ be the set of such products with weight i. For a product $\beta_{j,t} = \prod \overline{x}_{i,s}^{\alpha_{i,s}}$ Alperin defines a complementary element, $\beta_{j,t}^c = \prod \overline{x}_{i,s}^{p-1-\alpha_{i,s}}$. We further define a coefficient $c_{i,t} = \prod \binom{p-1}{\alpha_{i,s}}$. The element $\beta_{\lambda-1,1} = \prod \overline{x}_{i,s}^{p-1} = \|G\|$ is the generator of the socle of $\mathbb{F}G$, and has weight $\lambda - 1 = \ell\ell(\mathbb{F}G) - 1$. The sum of the weights of $\beta_{i,t}$ and $\beta_{i,t}^c$ is $\lambda - 1$.

The following result will be used in section 3 when we determine the tensor decomposition of $\mathbb{F}G$ by uniserial modules.

Lemma 1.11. *With the notation just defined, if M and N are left $\mathbb{F}G$-modules, $m \in \mathrm{Soc}^\mu(M)$, $n \in \mathrm{Soc}^\nu(N)$, and $\mu + \nu - 1 = \lambda = \ell\ell(\mathbb{F}G)$, then*

$$\|G\| \cdot m \otimes n = \sum_t c_{\mu-1,t}\beta_{\mu-1,t} m \otimes \beta_{\mu-1,t}^c n.$$

Proof. We use the fact that $\|G\| = \prod \overline{x}_{i,s}^{p-1}$, where we have specified the order of multiplication. We start by multiplying $m \otimes n$ by the last term in the product. We have $\overline{x}_{i,s} \cdot m \otimes n = (x_{i,s} - 1) \cdot m \otimes n = \overline{x}_{i,s} m \otimes n + m \otimes \overline{x}_{i,s} n + \overline{x}_{i,s} m \otimes \overline{x}_{i,s} n$. Since $\overline{x}_{i,s} \in \mathrm{Rad}^i(\mathbb{F}G)$, we have $\overline{x}_{i,s} m \in \mathrm{Soc}^{\mu-i}(M)$ and $\overline{x}_{i,s} n \in \mathrm{Soc}^{\nu-i}(N)$.

We proceed by multiplying by $\overline{x}_{i,s}^{p-1}$. At each stage, we have a sum of elements which is multiplied on the left by $\overline{x}_{i,s}$. Each element gives a sum of three further elements, where $\overline{x}_{i,s}$ is applied on the left tensor factor, the right factor, or both factors. Thus

$$\overline{x}_{i,s}^{p-1} \cdot m \otimes n = S_1 + \sum_{j=0}^{p-1} \binom{p-1}{j} \overline{x}_{i,s}^j m \otimes \overline{x}_{i,s}^{p-1-j} n,$$

where S_1 is the sum of all elements for which the multiplication was applied to both tensor factors at least once.

We continue this process for the entire product, and obtain

$$\|G\| \cdot m \otimes n = S + \sum_{\gamma,t} c_{\gamma,t}\beta_{\gamma,t} m \otimes \beta_{\gamma,t}^c n,$$

where S is the sum of all elements for which at least on of the factors of $\|G\|$ was applied to both tensor factors at the same stage.

Now m is an element of $\mathrm{Soc}^{\mu}(M)$. Thus if m is multiplied by an element of $\mathrm{Rad}^{\mu}(\mathbb{F}G)$, the result is zero. Likewise, if n is multiplied by an element of $\mathrm{Rad}^{\nu}(\mathbb{F}G)$, the result is zero. The weights of $\beta_{\gamma,t}$ and $\beta^c_{\gamma,t}$ are γ and $\lambda-\gamma-1$, respectively. The sum of these weights is $\lambda-1=\mu-1+\nu-1$. The weight of an element is the minimum power of the radical in which it lies. Thus S must be zero, as well as all other summands except for the terms $c_{\mu-1,t}\beta_{\mu-1,t}m\otimes\beta^c_{\mu-1,t}n$. □

Lemma 1.12. *If m and n are elements of $\mathbb{F}G$, and $\mathrm{Rad}(\mathbb{F}Gn)\subseteq\mathrm{Rad}^2(\mathbb{F}Gm)$ then $\mathrm{Rad}(\mathbb{F}Gm)=\mathrm{Rad}(\mathbb{F}G(m+n))$.*

Proof. Given $g\in G$, we know that $\overline{g}n=(g-1)n$ is an element of $\mathrm{Rad}(\mathbb{F}Gn)$. Let $a_1=-\overline{g}$. As $\overline{g}n\in\mathrm{Rad}^2(\mathbb{F}Gm)$, there is an element $a_2\in\mathrm{Rad}^2(\mathbb{F}G)$ such that $a_2m=-a_1n=\overline{g}n$. We find elements $a_i\in\mathrm{Rad}^i(\mathbb{F}G)$ such that $a_{i+1}m=-a_in$. We set $a_i=0$ for all $i\geq\mu=\ell\ell(\mathbb{F}Gm)$. We find that

$$\sum_{i=2}^{\mu}a_i(m+n)=\sum_{i=2}^{\mu}a_in-\sum_{i=1}^{\mu-1}a_in=a_\mu n-a_1n=\overline{g}n,$$

which is an element of $\mathrm{Rad}^2(\mathbb{F}G(m+n))$. Thus $\mathrm{Rad}^1(\mathbb{F}Gn)\subseteq\mathrm{Rad}^2(\mathbb{F}G(m+n))$. The submodule $\mathrm{Rad}^1(\mathbb{F}G(m+n))$ is generated by elements of the form $(g-1)(m+n)=\overline{g}(m+n)=\overline{g}m+\overline{g}n$. But $\overline{g}n\in\mathrm{Rad}^2(\mathbb{F}G(m+n)$, so the radical is generated by all elements of the form $\overline{g}m$. This is also the radical of $\mathbb{F}Gm$. □

If we have one tensor decomposition of $\mathbb{F}G$, we now show a way to find other tensor decompositions.

Lemma 1.13. *If m and m_1 are elements of $\mathbb{F}G$, $\mathrm{Rad}(\mathbb{F}Gm_1)\subseteq\mathrm{Rad}^2(\mathbb{F}Gm)$, $\mathbb{F}G\cong\mathbb{F}Gm\otimes N$, and $\ell\ell(\mathbb{F}Gm)+\ell\ell(N)-1=\ell\ell(\mathbb{F}G)$ then $\mathbb{F}G\cong\mathbb{F}G(m+m_1)\otimes N$.*

Proof. By Lemma 1.10 we see that there is an element n that generates N. Lemma 1.9 implies that there is an element $m'\otimes n'$ such that $\|G\|m'\otimes n'\neq 0$. As m and n generate $\mathbb{F}Gm$ and N, we may assume that $m'=m+rm$ and $n'=n+sn$, where r and s are elements of $\mathrm{Rad}(\mathbb{F}G)$, as non-zero scalar multiples are irrelevant.

Let $\mu = \ell\ell(\mathbb{F}Gm)$, and $\nu = \ell\ell(N)$. We must have m and m' in $\mathrm{Soc}^{\mu}(\mathbb{F}Gm)$ and n and n' in $\mathrm{Soc}^{\nu}(N)$. By Lemma 1.11 we see that

$$\|G\| \cdot m \otimes n = \sum_t c_{\mu-1,t}\beta_{\mu-1,t}m \otimes \beta^c_{\mu-1,t}n.$$

Replacing m and n by m' and n' respectively, we see that $\beta_{\mu-1,t}(m + rm) = \beta_{\mu-1,t}m$, and $\beta^c_{\mu-1,t}(n + sn) = \beta^c_{\mu-1,t}n$, as rm and sn are not in a high enough power of the socle to survive. Thus we may assume that $m' = m$, and $n' = n$. Any choice of generators for the two modules would suffice.

We now substitute $m + m_1$ for m. We have $\beta_{\mu-1,t}(m + m_1) = \beta_{\mu-1,t}m + \beta_{\mu-1,t}m_1$. But $m_1 \in \mathrm{Soc}^{\mu-1}(\mathbb{F}G)$, so this term is just $\beta_{\mu-1,t}m$. We have $\|G\| \cdot (m + m_1) \otimes n = \|G\| \cdot m \otimes n \neq 0$. By Lemma 1.12, we have $\mathrm{Rad}(\mathbb{F}Gm) = \mathrm{Rad}(\mathbb{F}G(m+m_1))$. Since these modules are both cyclic, and have the same radical, they must both have the same dimension. Again, using Lemma 1.9, we see that $\mathbb{F}G \cong \mathbb{F}G(m + m_1) \otimes N$. $\square$

We next present a result relating the radical series to the commutativity within $\mathbb{F}G$. This result will be useful to us in specific calculations involving the action of generators of $\mathrm{Rad}(\mathbb{F}G)$ on modules.

For the following lemma, G need not be a p-group.

Lemma 1.14. *If G is a group, H is a subgroup of index p, and g is an element of G not in H, then $\|G\| = (g - 1)^{p-1}\|H\|$ in a field of characteristic p.*

Proof. As g is outside of H, the cosets of H in G are $\{g^i H \mid 0 \leq i < p\}$. The element $(g - 1)^p = g^p - 1$ in $\mathbb{F}_pG$, so $(g - 1)^{p-1}$ is the sum of all powers less than p of g. Thus $(g - 1)^{p-1}\|H\|$ is the sum of all elements of all cosets of $\|H\|$ in G. $\square$

Corollary 1.15. *For the p-group G with Jennings series $G = \kappa_1 \supseteq \kappa_2 \supseteq \ldots$, and the set $\{x_{i,s} \mid 1 \leq i < n, 1 \leq s \leq d_i\}$ as described above, we may write*

$$\|\kappa_i\| = \prod_{s=1}^{d_i} \bar{x}_{i,s}^{p-1}\|\kappa_{i+1}\|,$$

and $\|\kappa_{i+1}\|$ is in the center of $\mathbb{F}\kappa_i$.

Proof. The group κ_i/κ_{i+1} is an elementary abelian group with generators $\{x_{i.s}\kappa_{i+1} \bigm| 1 \leq s \leq d_i\}$. We may thus use the lemma iteratively to obtain the first result. Also, we know that κ_{i+1} is normal in κ_i, so

$$x_{i,s}\|\kappa_{i+1}\| = \|x_{i,s}\kappa_{i+1}\| = \|\kappa_{i+1}x_{i,s}| = \|\kappa_{i+1}\|x_{i,s}.$$

$\square$

Corollary 1.16. *The center of $\mathbb{F}G$ contains* $\operatorname{Soc}^2(\mathbb{F}G)$.

Proof. From the previous corollary we have $\|\kappa_2\|$ is contained in the center of $\mathbb{F}G$. As κ_2 is the Frattini subgroup of G, we know that

$$gh\|\kappa_2\| = hg\ g^-1h^-1gh\|\kappa_2\| = hg\|\kappa_2\|,$$

since the Frattini subgroup contains the commutator subgroup. By induction, we find that $\|\kappa_l\| = \prod \bar{x}_{i,s}^{\alpha_{i,s}}$, where the product is taken over all i and s, and $\alpha_{i,s} = 0$ when $i < l$, and $p-1$ when $i \geq l$. Thus $\|\kappa_2\| = \prod \bar{x}_{i,s}^{\alpha_{i,s}}$, where $\alpha_{1,s} = 0$ for all s, and $\alpha_{i,s} = p-1$ for all $i \geq 2$ and all s.

A basis for $\operatorname{Soc}^2(\mathbb{F}G)$ is the set of elements $\{\beta_{j,t} \bigm| j \geq \ell\ell(\mathbb{F}G) - 2\}$. Let $\beta_{j,t} = \prod \bar{x}_{i,s}^{\alpha_{i,s}}$ be an element of weight $j = \ell\ell(\mathbb{F}G) - 2$. In order to have weight j, we need $\alpha_{i,s} = p-1$ except for one exponent, $\alpha_{1,t} = p-2$, for some t. We see that $\|\kappa_2\|$ is a factor of $\beta_{j,t}$, and we may rewrite

$$\beta_{j,t} = \prod_s \bar{x}_{1,s}^{p-1-\delta_{s,t}}\ \|\kappa_2\|,$$

where $\delta_{s,t} = 1$ when $s = t$, and 0 otherwise. We know that a full set of generators for G is $\{x_s = \bar{x}_s{+}1\}$. As elements commute modulo $\|\kappa_2\|$, we see that $g\beta_{j,t} = \beta_{j,t}g$ for all g in G, and since the $\beta_{j,t}$ form a basis for $\operatorname{Soc}^2(\mathbb{F}G)$, this establishes the result. $\square$

For each maximal subgroup H of G, choose an element g_H of G that is not in H. Let $S_H = \sum_{i=0}^{p-1} ig_H^i\|H\|$. We recall that $\mathbb{F}_p$ is the field of p elements.

Theorem 1.17. *The complete list of elements of* $\mathrm{Soc}^2(\mathbb{F}_pG)$ *is*

$$\{f\|G\| \bigm| f \in \mathbb{F}_p\} \cup \{fg_H^iS_H \bigm| H \text{ is maximal in } G, 0 \neq f \in \mathbb{F}, 0 \leq i < p\}.$$

Proof. First we show that the elements in question are indeed elements of $\mathrm{Soc}^2(\mathbb{F}_pG)$. We know that $\mathrm{Soc}(\mathbb{F}_pG)$ is spanned by $\|G\|$. Thus the first set of elements is contained in $\mathrm{Soc}(\mathbb{F}_pG)$. Let H be a maximal subgroup of G. As G is a p-group, every maximal subgroup of G is normal with index p. Thus any element of G may be written in the form g_H^jh for some element $h \in H$, and some j between 0 and $p-1$. As $G/H \cong C_p$, we have $g_H^jhg_H^i\|H\| = g_H^{i+j}\|H\|$, and

$$\begin{aligned} g_H^jhS_H &= \sum_{i=0}^{p-1} ig_H^{i+j}\|H\| \\ &= \sum_{i=0}^{p-1}(i+j)g_H^{i+j}\|H\| - j\sum_{i=0}^{p-1} g_H^{i+j}\|H\| \\ &= S_H - j\|G\|. \end{aligned}$$

Thus $(g_H^jh - 1)S_H = -j\|G\| \in \mathrm{Soc}(\mathbb{F}_pG)$. As $\mathrm{Rad}(\mathbb{F}_pG)$ is generated by elements of the form $g-1$, we see that $S_H \in \mathrm{Soc}^2(\mathbb{F}_pG)$, for every maximal subgroup H of G. As $\mathrm{Soc}^2(\mathbb{F}_pG)$ is an ideal, we also have $fg_H^iS_H \in \mathrm{Soc}^2(\mathbb{F}_pG)$, for all $f \in \mathbb{F}_p$, and $0 \leq i < p$.

We show that no element is listed more than once in the statement of the theorem. Clearly the elements of the first set are distinct, and different from all elements of the second set. Assume that the element $f_1g_H^iS_H = f_2g_K^jS_K$ for some elements f_1 and f_2 of $\mathbb{F}_p$, and some maximal subgroups H and K. In $f_1g_H^iS_H$, every element of a coset gH has the same coefficient. Thus the subgroup H may be determined. This indicates that $H = K$. Only one of the cosets of H has coefficient 0 in $f_1g_H^iS_H$. This coset must be g_H^i, as $\|H\|$ is the coset in S_H with coefficient 0. This forces i and j to be equal. The two expressions of the same element both are the same multiple of the element $g_H^iS_H$, so $f_1 = f_2$. Thus the elements of the second set are distinctly listed.

We then count the number of elements of $\mathrm{Soc}^2(\mathbb{F}_pG)$, and find it to be the same as the number of elements listed in the statement of the theorem. We know that the Frattini subgroup of G is $\kappa_2(G)$. The elementary abelian group G/κ_2 is isomorphic to a vector space of dimension d_1. The maximal subgroups of G are generated by maximal subspaces of G/κ_2. The number of maximal subspaces of a vector space is equal to the number of projective lines in that subspace. The number of projective lines in G/κ_2 is $(p^{d_1} - 1)/(p - 1)$. Thus, the number of non-zero elements of the form $fg_H^iS_H$ is $(p-1)p(p^{d_1} - 1)/(p-1) = p^{d_1+1} - p$. The number of elements of $\mathrm{Soc}(\mathbb{F}_pG)$ is p. Thus, the number of elements listed is p^{d_1+1}.

The vector space $\mathrm{Soc}^2(\mathbb{F}_pG)$ has a basis $\{\bar{x}_{1,s}^c \bigm| 1 \leq s \leq d_1\} \cup \{\|G\|\}$. This gives $\dim(\mathrm{Soc}^2(\mathbb{F}_pG)) = d_1 + 1$, and the number of elements in $\mathrm{Soc}^2(\mathbb{F}_pG)$ is p^{d_1+1}. $\square$

The particular case where $p = 2$ has a simpler form.

Corollary 1.18. *If G is a 2-group then the elements of $\mathrm{Soc}^2(\mathbb{F}_2G)$ consists of the elements*

$$\{0, \|G\|\} \cup \{\|H\|, \|H\|^c \bigm| H \text{ is maximal in } G\},$$

where $\|H\|^c$ refers to the complement of $\|H\|$, i.e. $\|G\| - \|H\|$.

Proof. The field $\mathbb{F}_2$ has only 2 elements, 1 and 0. Given a maximal subgroup H of G, let g be an element of G not in H. We see that $S_H = g\|H\|$, and $gS_H = \|H\|$. In the statement of the theorem, we see that f and i may only be 0 or 1, and the result follows. $\square$

§2 Permutation Module Decompositions

In this section, we look at decompositions of $\mathbb{F}G$ into the tensor product $M \otimes N$, where both M and N are permutation modules. By restricting our attention to permutation modules, we are able to obtain cleaner results than appear to hold for general tensor decompositions. A *permutation module* for $\mathbb{F}G$ is a module which has a basis permuted by G. The first lemma is well known.

Lemma 2.1. *Let M be a left $\mathbb{F}G$ permutation module with G acting transitively on the basis B of M. If $H = \mathrm{Stab}_G(b)$, for some basis element $b \in B$, then $M \cong \mathbb{F}G\|H\| \cong \mathbb{F}[G/H]$.*

Proof. Let T be a left transversal for the subgroup H in G. Let 1 represent the coset $1H$. The basis B for M is the G orbit of b, i.e., $B = \{gb \mid g \in G\} = \{tb \mid t \in T\}$. Likewise, a basis for $\mathbb{F}G\|H\|$ is $\{t\|H\| = \|tH\| \bigm| t \in T\}$, and a basis for $\mathbb{F}[G/H]$ is $\{tH \mid t \in T\}$. Make the relation between basis elements,

$$tb \leftrightarrow t\|H\| \leftrightarrow tH.$$

This identification preserves the action of G, and the isomorphism is found by extending linearly to the entirety of the modules. $\square$

Theorem 2.2. *Let H and K be subgroups of G and let M and N be the left $\mathbb{F}G$-modules generated by $\|H\|$ and $\|K\|$, respectively. Then $\mathbb{F}G \cong M \otimes N$ if and only if $G = HK$ and $H \cap K = 1$.*

Proof. We give a proof in the case where G is a p-group and $\mathbb{F}$ has characteristic p. If $HK \neq G$ then either $H \cap K \neq 1$, or $|H||K| < |G|$. If $|H||K| < |G|$, then the dimension of $\mathbb{F}G\|H\| \otimes \mathbb{F}G\|K\|$ is $|H||K|$, and is less than $|G|$, which is the dimension of $\mathbb{F}G$.

Let $Q = H \cap K$. The element $\|H\| \otimes \|K\|$ is a generator of $\mathbb{F}G\|H\| \otimes \mathbb{F}G\|K\|$. We have,

$$\begin{aligned}
\|Q\| \cdot \|H\| \otimes \|K\| &= \sum_{q \in Q} q\|H\| \otimes q\|K\| \\
&= \sum_{q \in Q} \|H\| \otimes \|K\| \\
&= |Q|\|H\| \otimes \|K\|.
\end{aligned}$$

If $Q \neq 1$, then this product is equal to 0, as p divides the order of Q. However, $\|Q\|1 = \|Q\| \neq 0$ in $\mathbb{F}G$. If an element of $\mathbb{F}G$ other than zero annihilates a generator of a module, then that module must not be isomorphic to the regular representation.

Bases for M and N are $\{\|kH\| : k \in K\}$ and $\{\|hK\|\} : h \in H\}$, respectively. (Each basis element is the sum of elements of a particular coset of the group.) A basis for $M \otimes N$ is then

$$\{\|kH\| \otimes \|hK\| : k \in K, h \in H\}.$$

The group G permutes these basis elements. The total number of basis elements is $|G|$. The stabilizer of the element $\|H\| \otimes \|K\|$ must be in both H and K, so $\mathrm{Stab}_G(\|H\| \otimes \|K\|) = 1$. This shows that the size of the orbit of $\|H\| \otimes \|K\|$ is $|G|$, so the orbit must be the entire basis. The basis is permuted regularly by G, so $M \otimes N \cong \mathbb{F}G$. □

Corollary 2.3. *Let $\mathbb{F}$ be a field of characteristic p. Let G be a group, which is not necessarily a p-group. Let H and K be subgroups of G with trivial intersection. Let M and N be the $\mathbb{F}G$-modules generated by $\|H\|$ and $\|K\|$, respectively. Then $\mathbb{F}G \cong M \otimes N$ if and only if $G = HK$.*

Proof. The only part of the proof of the theorem which uses the fact that G is a p-group involves the order of the intersection of H and K. If we insist that this intersection is trivial, the rest of the proof holds. □

We comment that an analogous, but more complicated statement may be made concerning decompositions of $\mathbb{F}G$ as a tensor product of three or more permutation modules.

Let H and K be subgroups of the group G. Then K is a *complement* of H if $HK = G$, and $H \cap K = 1$.

If α is an automorphism of the group G, and K is a complement to H, it is a simple matter to show that $\alpha(K)$ is also a complement of $\alpha(H)$ in G. It is useful to know that if α is an inner automorphism, then H also has the complement $\alpha(K)$.

Lemma 2.4. *Let H and K be subgroups in G. If K is a complement of H, and g is any element of G, then ${}^gK$ is also a complement of H.*

Proof. We may write $g = hk$ for some elements h of H and k of K. Now $H\,{}^gK = HhkKk^{-1}h^{-1} = HKh^{-1} = Gh^{-1} = G$. Since $|H \cap {}^gK||H\,{}^gK| = |H||{}^gK| = |G|$, we have $|H \cap {}^gK| = 1$. $\square$

§3 Tensor Decompositions of $\mathbb{F}G$, where $|G| = 16$

In this paper we have been paying particular attention to the groups of order 16 which may not be non-trivially written as the direct product of two subgroups. The calculations we are about to perform depend heavily on an examination of the subgroup lattices of these groups, which we take to be readily available for inspection either from [H&S] or by calculations with the program [Sc]. We also rely on the lists of uniserial modules for each group ring, which were obtained in Chapter 2.

For each group G of order 16, we consider non-trivial tensor decompositions $\mathbb{F}G \cong M \otimes N$. If $\mathbb{F}G \cong M \otimes N$, then $\dim(M)\dim(N) = 16$ and $\ell\ell(\mathbb{F}G) \leq \ell\ell(M) + \ell\ell(N) - 1$ by Lemma 1.4. By symmetry of the tensor decomposition, we need only consider cases where $\dim(M) \leq \dim(N)$, so we must have $\dim(M) = 2$ and $\dim(N) = 8$ or $\dim(M) = \dim(N) = 4$. By Lemma 1.10 we need only consider cyclic modules M and N. If M is cyclic and $\dim(M) = 2$, then M is uniserial, and $\ell\ell(M) = 2$.

We first consider tensor decompositions in which both factors are permutation modules. If such a decomposition exists, Theorem 2.2 tells us that there must be a pair of complementary subgroups H and K in G for which M is generated by $\|H\|$ and N is generated by $\|K\|$. By symmetry of the tensor product, we may take H to be the larger of the two subgroups. The dimension of the module is equal to the index of the subgroup in the group, so $|H| = 8$ and $|K| = 2$, or $|H| = |K| = 4$. With these choices, $G = HK$ only when $H \cap K = 1$, so we need only consider pairs which have trivial intersections. By Lemma 2.4 we need only consider representatives of conjugacy classes for both H and K.

For each group the full subgroup lattice is given. The larger subgroups are listed above the smaller subgroups. Each subgroup of a given order is listed at the same level. A solid line between two subgroups indicates that the smaller subgroup is contained in the larger subgroup, and there is no intermediate subgroup. The diagram also indicates the orbits of subgroups under the actions of the automorphisms of the group. A triple dotted line between two entries indicates that the two subgroups are conjugates. A single dotted line between the listings of two subgroups indicates that there is an outer automorphism of G that maps one subgroup to the other. If there is an automorphism that maps one subgroup to another subgroup, then there is a path of dotted lines connecting the listings of the two subgroups.

We start with the groups G of order 16 for which $\mathbb{F}G$ has Loewy length 7. These are the groups G_1 and $(2 \times 4).2$. In both cases, the Poincaré polynomial associated to $\mathbb{F}G$ is $P_{\mathbb{F}G}(t) = 1 + 2t + 3t^2 + 4t^3 + 3t^4 + 2t^5 + t^6$. Assume $\mathbb{F}G \cong M \otimes N$. If $\dim(M) = 2$ then $7 \leq 2 + \ell\ell(N) - 1 \leq 1 + 6 = 7$, as $\ell\ell(N) < \ell\ell(\mathbb{F}G)$. Thus $\ell\ell(N) = 6$ and $\dim(N) = 8$. The Poincaré polynomials for M and N are $P_M(t) = 1+t$ and $P_N(t) = 1+t+2t^2+2t^3+t^4+t^5$. If $\dim(M) = \dim(N) = 4$ then $7 \leq \ell\ell(M)+\ell\ell(N)-1 \leq 4+4-1 = 7$, forcing equality, and both M and N must be uniserial. Their corresponding Poincaré polynomials are both $P(t) = 1+t+t^2+t^3$.

Given M and N, we wish to determine if $\mathbb{F}G \cong M \otimes N$. We use Lemma 1.9 to determine this. We only consider cases where $\dim(M)\dim(N) = 16$, and either $\ell\ell(M) = 2$ and $\ell\ell(N) = 6$ or $\ell\ell(M) = \ell\ell(N) = 4$. If we can show that the product $\|G\| \cdot M \otimes N \neq 0$, then we know that $\mathbb{F}G \cong M \otimes N$. Let m and n be generators of M and N respectively. By Lemma 1.11 of Chapter 1, the choice of generators does not matter. The product $\|G\| \cdot m \otimes n$ is non-zero if and only if the product $\|G\| \cdot M \otimes N$ is non-zero. We use Lemma 1.11 to find $\|G\| \cdot m \otimes n$.

In both groups that we are considering, we have chosen bases for $\mathbb{F}G$ consisting of elements of the form $\beta_{i+j,i} = \bar{x}^i\bar{y}^j$, where $0 \leq i,j \leq 3$. The complementary element is $\beta^c_{i+j,i} = \bar{x}^{3-i}\bar{y}^{3-j}$. Note that the characteristic of $\mathbb{F}$ is 2, so $c_{i+j,i} = 1$.

If $\dim(M) = 2$ and $\dim(N) = 8$, we have $\mu = 2$ and

$$\begin{aligned}\|G\| \cdot m \otimes n &= \sum_t c_{\mu-1,t}\beta_{\mu-1,t}m \otimes \beta^c_{\mu-1,t}n \\ &= \bar{x}m \otimes \bar{x}^2\bar{y}^3n + \bar{y}m \otimes \bar{x}^3\bar{y}^2n.\end{aligned}$$

If $\dim(M) = \dim(N) = 4$, we have $\mu = 4$ and

$$\|G\| \cdot m \otimes n = \bar{x}^3m \otimes \bar{y}^3n + \bar{x}^2\bar{y}m \otimes \bar{x}\bar{y}^2n + \bar{x}\bar{y}^2m \otimes \bar{x}^2\bar{y}n + \bar{y}^3m \otimes \bar{x}^3n.$$

We now focus our attention on the group

$$G_1 = \left\langle x, y \mid x^4 = y^4 = (xy)^2 = 1, \{x^2, y^2\} \subset Z \right\rangle,$$

where Z is the center of the group. We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1.

We present the diagram for the full subgroup lattice of G_1:

Theorem 3.1. *The group ring $\mathbb{F}G_1$ is decomposable into the tensor product of permutation modules, $\mathbb{F}G\|H\| \otimes \mathbb{F}G\|K\|$. Up to interchanging H and K, these are the only such pairs of subgroups.*

a) *The subgroup H is either $\langle x, y^2 \rangle$ or $\langle x^2, y \rangle$ and K is any one of the non-central subgroups of order 2;*

b) *The subgroup H is either $\langle x \rangle$ or $\langle xy^2 \rangle$ and K is one of the four subgroups $\langle xy, x^2y^2 \rangle$, $\langle x^{-1}y, x^2y^2 \rangle$, $\langle xy, y^2 \rangle$, or $\langle x^{-1}y, y^2 \rangle$;*

b$'$) *The subgroup H is either $\langle y \rangle$ or $\langle x^2y \rangle$ and K is one of the four subgroups $\langle xy, x^2y^2 \rangle$, $\langle x^{-1}y, x^2y^2 \rangle$, $\langle xy, x^2 \rangle$, or $\langle xy^{-1}, x^2 \rangle$;*

c) *The subgroup H is either $\langle x \rangle$ or $\langle xy^2 \rangle$ and K is either $\langle y \rangle$ or $\langle x^2y \rangle$.*

Note that the complementary pairs of b$'$) may be obtained by applying the automorphism that exchanges x and y to the complementary pairs of b).

Proof. By observing the diagram, we determine all complementary pairs of subgroups in G_1. By Theorem 2.2 we see that these subgroup pairs correspond with all possible tensor decompositions of $\mathbb{F}G_1$ by permutation modules. $\square$

We now list, up to isomorphism, all permutation modules of $\mathbb{F}G_1$, identifying those that are uniserial on our list of all uniserial modules. To every subgroup H of G_1, there is a module $\mathbb{F}G_1\|H\|$. We list the subgroups in the order of decreasing size, with conjugate subgroups listed together. For each subgroup H, we list a generator of $\mathbb{F}G_1\|H\|$, as described in Chapter 2, section 1. The generators of the form $E^{(6)}_{r,s,t,u,v}$ are described after the listing.

- $\langle x, y\rangle = G_1$ corresponds to $A^{(1)}$.
- $\langle x^2, y^2, xy\rangle \cong C_2^3$ corresponds to $A^{(2)}_1$.
- $\langle y, x^2\rangle \cong C_2 \times C_4$ corresponds to $A^{(2)}_0$.
- $\langle x, y^2\rangle \cong C_2 \times C_4$ corresponds to $B^{(2)}$.
- $\langle xy, x^2y^2\rangle \cong C_2^2$ corresponds to $A^{(4)}_{1,1,1} = C^{(4)}_{1,1,0,1}$.
- $\langle x^{-1}y, x^2y^2\rangle \cong C_2^2$ corresponds to $A^{(4)}_{1,0,0} = C^{(4)}_{1,0,0,0}$.
- $C_2^2 \cong \langle xy, x^2\rangle$ and $\langle xy^{-1}, x^2\rangle$ correspond to $D^{(4)}_{0,0,0}$ and $D^{(4)}_{1,0,1}$.
- $C_2^2 \cong \langle xy, y^2\rangle$ and $\langle x^{-1}y, y^2\rangle$ correspond to $C^{(4)}_{0,1,0,1}$ and $C^{(4)}_{0,0,0,0}$.
- $\langle x^2, y^2\rangle \cong C_2^2$ corresponds to $\bar{x}^2\bar{y}^2$, which is not a uniserial module generator.
- $C_4 \cong \langle x\rangle$ and $\langle xy^2\rangle$ correspond to $B^{(4)}_{1,0}$ and $B^{(4)}_{0,0}$.
- $C_4 \cong \langle y\rangle$ and $\langle x^2y\rangle$ correspond to $A^{(4)}_{0,0,0}$ and $A^{(4)}_{0,1,0}$.
- $C_2 \cong \langle xy\rangle$ and $\langle x^{-1}y^{-1}\rangle$ correspond to $E^{(6)}_{1,0,0,0,0}$ and $E^{(6)}_{0,1,0,0,0}$.
- $C_2 \cong \langle xy^{-1}\rangle$ and $\langle x^{-1}y\rangle$ correspond to $E^{(6)}_{1,1,0,0,0}$ and $E^{(6)}_{0,0,0,0,0}$.
- $\langle x^2\rangle \cong C_2$ corresponds to $\bar{x}^2$, which is not a uniserial module generator.
- $\langle y^2\rangle \cong C_2$ corresponds to $\bar{y}^2$, which is not a uniserial module generator.
- $\langle x^2y^2\rangle \cong C_2$ corresponds to $\bar{x}^2+\bar{y}^2$, which is not a uniserial module generator.
- $\langle 1\rangle$ generates all of $\mathbb{F}G_1$.

We now find all modules M and N such that $\mathbb{F}G_1 \cong M\otimes N$. We start with the case where $\dim(M) = 2$ and $\dim(N) = 8$. We need to find all cyclic submodules N of $\mathbb{F}G_1$ with Poincaré polynomial $P_N = 1 + t + 2t^2 + 2t^3 + t^4 + t^5$. We use an algorithm similar to that described in Chapter 1, Section 2. We sketch the use of the algorithm, although the actual calculations are performed by computer, and are not shown here. We use the basis from Chapter 2, Section 1 for $\mathbb{F}G_1$. We start with an arbitrary element of $\mathrm{Soc}^6(\mathbb{F}G_1)$,

$$E^{(6)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & b & & c & & \\ & d & & e & & f & \\ g & & h & & i & & j \\ & k & & l & & m & \\ & & n & & p & & \\ & & & q & & & \end{array} .$$

Let $N = \mathbb{F}G_1 E^{(6)}$. We need $E^{(6)} \notin \mathrm{Soc}^5(\mathbb{F}G_1)$. This implies that b and c can not both be zero. Consider the map from $\mathrm{Rad}(N)$ to $\mathrm{Rad}(N)/\mathrm{Rad}^2(N)$, where we list the elements in terms of the images of the basis elements $\bar{x}^2$, $\bar{x}\bar{y}$, and $\bar{y}^2$ of $\mathbb{F}G_1$. Using the multiplications that are given in Chapter 2, Section 1, we find that the images of $\bar{x}E^{(6)}$ and $\bar{y}E^{(6)}$ are $(b, c, 0)$ and $(b, b, b + c)$ respectively. Since we need $\dim(\mathrm{Rad}(N)/\mathrm{Rad}^2(N)) = 1$, we see that both b and c must be equal. By rescaling, we choose $b = c = 1$.

We find the set of elements $\bar{x}^i\bar{y}^j E^{(6)}$, and perform a Gram-Schmidt type of process on the set. For each of the entries b, d, g, i, k, l, n, and q in the original arbitrary element $E^{(6)}$, we find an element in our resulting set that has a 1 in the corresponding position, and only zeros in all prior entries. We then conclude that a distinguished generator of our module should have a zero in each of these positions, other than the position designated by b. After setting the values of these variables as indicated, we find that there remain two vectors that are not in the form of a distinguished generator. The module is either an 8-dimensional or a 10-dimensional vector space, depending on the values of these two vectors. As we desire an 8-dimensional space, we require these two remaining vectors to be zero. The two vectors are zero precisely when $j = e + ef + e^2 + f + h$. We thus set j to this value. The distinguished generators of $\mathbb{F}G_1$ which generate 8-dimensional modules of Loewy length 6 are of the form

$$E^{(6)}_{r,s,t,u,v} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & 1 & & 1 & & \\ & \cdot & & r & & s & \\ \cdot & & t & & \cdot & & r + rs + r^2 + s + t \\ & \cdot & & \cdot & & u & \\ & & \cdot & & v & & \\ & & & \cdot & & & \end{array}$$

The isomorphism bucket for the module generated by such an element is

$$\left|\begin{array}{c} E^{(6)}_{r,s,t,u,v} \\ \\ \scriptstyle r+s=\alpha,\, r^2t+v=\beta \\ \hline \end{array}\right|$$

$$\alpha, \beta, t \in \mathbb{F}$$

In order to find all modules M and N such that $\mathbb{F}G_1 \cong M \otimes N$, we need to find the products $\|G_1\| \cdot m \otimes n$ that are non-zero, where m generates M, and n generates N. As we have seen, if M is two dimensional, and N is eight dimensional, then we need to find $\bar{x}m$, $\bar{y}m$, $\bar{x}^2\bar{y}^3n$, and $\bar{x}^3\bar{y}^2n$. We present these products in the following multiplication table:

	$E^{(6)}_{r,s,t,u,v}$	$A^{(2)}_r$	$B^{(2)}$
$\bar{x}^2\bar{y}^3$	$A^{(1)}$	0	0
$\bar{x}^3\bar{y}^2$	$A^{(1)}$	0	0
$\bar{x}$	$-$	$A^{(1)}$	0
$\bar{y}$	$-$	$rA^{(1)}$	$A^{(1)}$

Table of elements of $(IG_1)u$ and $(IG_1)^5u$

Any entry of $-$ in this table is an irrelevant product. We see that

$$\begin{aligned} \|G_1\|A^{(2)}_\rho \otimes E^{(6)}_{r,s,t,u,v} &= \bar{x}A^{(2)}_\rho \otimes \bar{x}^2\bar{y}^3E^{(6)}_{r,s,t,u,v} + \bar{y}A^{(2)}_\rho \otimes \bar{x}^3\bar{y}^2E^{(6)}_{r,s,t,u,v} \\ &= (1+\rho)A^{(1)} \otimes A^{(1)}, \end{aligned}$$

and this is non-zero whenever $\rho \neq 1$. Likewise, if we substitute $B^{(2)}$ in the previous equation in place of $A^{(2)}$, we find that $\|G_1\|B^{(2)} \otimes E^{(6)}_{r,s,t,u,v} = A^{(1)} \otimes A^{(1)} \neq 0$. We have found the tensor decompositions $\mathbb{F}G_1 \cong M \otimes N$ where $\dim(M) = 2$ and $\dim(N) = 8$. We restate these results in the following table. We list the possible values of m on the top, and the possible values of n on the side. The coefficient of $A^{(1)} \otimes A^{(1)}$ in the product $\|G_1\| \cdot m \otimes n$ is listed in the corresponding row and column. Any non-zero entry indicates that there is a tensor decomposition $\mathbb{F}G_1 \cong \mathbb{F}G_1m \otimes \mathbb{F}G_1n$.

	$A^{(2)}_{\rho}$	$B^{(2)}$
$E^{(6)}_{r,s,t,u,v}$	$1+\rho$	1

Table of coefficients for $A^{(1)} \otimes A^{(1)}$ in $\|G_1\| \cdot m \otimes n$

If M and N are both 4-dimensional, we need to find the products, $\bar{x}^3 u$, $\bar{x}^2\bar{y}u$, $\bar{x}\bar{y}^2 u$, and $\bar{y}^3 u$, where u is a generator of M or N. We have seen that both M and N need to be uniserial. We obtain the following multiplication table:

	$A^{(4)}_{r,s,t}$	$B^{(4)}_{s,t}$	$C^{(4)}_{s,t,u,v}$	$D^{(4)}_{s,t,u}$
$\bar{x}^3$	$A^{(1)}$	0	$A^{(1)}$	0
$\bar{x}^2\bar{y}$	$rA^{(1)}$	0	$A^{(1)}$	0
$\bar{x}\bar{y}^2$	$rA^{(1)}$	0	$sA^{(1)}$	$A^{(1)}$
$\bar{y}^3$	$r^2A^{(1)}$	$A^{(1)}$	$sA^{(1)}$	$A^{(1)}$

Table of elements of $(IG_1)^3$ u

We wish to determine all non-zero products of the form

$$\|G_1\| \cdot m \otimes n = \bar{x}^3 m \otimes \bar{y}^3 n + \bar{x}^2\bar{y}m \otimes \bar{x}\bar{y}^2 n + \bar{x}\bar{y}^2 m \otimes \bar{x}^2\bar{y}n + \bar{y}^3 m \otimes \bar{x}^3 n.$$

This table gives the coefficient of $A^{(1)} \otimes A^{(1)}$ in the product $\|G_1\| \cdot m \otimes n$.

	$A^{(4)}_{r,s,t}$	$B^{(4)}_{s,t}$	$C^{(4)}_{s,t,u,v}$	$D^{(4)}_{s,t,u}$
$A^{(4)}_{\rho,\sigma,\tau}$	$r^2+\rho^2$	1	$(s+\rho)(\rho+1)$	$1+\rho$
$B^{(4)}_{\sigma,\tau}$	1	0	1	0
$C^{(4)}_{\sigma,\tau,\upsilon,\nu}$	$(\sigma+r)(r+1)$	1	0	0
$D^{(4)}_{\sigma,\tau,\upsilon}$	$1+r$	0	0	0

Table of coefficients of $A^{(1)} \otimes A^{(1)}$ in $\|G\| \cdot m \otimes n$

This table describes all tensor decompositions of $\mathbb{F}G_1 \cong M \otimes N$, when both M and N are uniserial. A generator of each uniserial module of length 4 is listed on the top and on the side. If the entry corresponding to generators m and n is non-zero, then a tensor decomposition by those modules exists. For example, the entry corresponding to $A^{(4)}_{r,s,t}$ and $A^{(4)}_{\rho,\sigma,\tau}$ is $r^2 + \rho^2$. This indicates that if $r^2 + \rho^2 \neq 0$, (i.e., $r \neq \rho$), then $\mathbb{F}G_1 \cong \mathbb{F}G_1 A^{(4)}_{\rho,\sigma,\tau} \otimes \mathbb{F}G_1 A^{(4)}_{r,s,t}$. In this manner, we determine all tensor decompositions of $\mathbb{F}G_1$:

Theorem 3.2. *Up to isomorphism and rearrangement of factors, the following is the complete list of non-trivial tensor decompositions,* $\mathbb{F}G_1 \cong M \otimes N$*:*

a) $M \cong \mathbb{F}G_1 A^{(2)}_{\rho}$ *and* $N \cong \mathbb{F}G_1 E^{(6)}_{0,\alpha,t,0,\beta}$ *when* $\rho \neq 1$*;*
b) $M \cong \mathbb{F}G_1 B^{(2)}$ *and* $N \cong \mathbb{F}G_1 E^{(6)}_{0,\alpha,t,0,\beta}$*;*
c) $M \cong \mathbb{F}G_1 A^{(4)}_{\rho,\sigma,\tau}$ *and* $N \cong \mathbb{F}G_1 A^{(4)}_{r,s,t}$ *when* $\rho \neq r$*;*
d) $M \cong \mathbb{F}G_1 A^{(4)}_{\rho,\sigma,\tau}$ *and* $N \cong \mathbb{F}G_1 B^{(4)}_{s,t}$*;*
e) $M \cong \mathbb{F}G_1 A^{(4)}_{\rho,\sigma,\tau}$ *and* $N \cong \mathbb{F}G_1 C^{(4)}_{s,t,u,v}$ *when* $s \neq \rho$ *and* $\rho \neq 1$*, and if* $s = 1$ *then* $u \neq 0$*;*
f) $M \cong \mathbb{F}G_1 A^{(4)}_{\rho,\sigma,\tau}$ *and* $N \cong \mathbb{F}G_1 D^{(4)}_{s,t,u}$ *with* $\rho \neq 1$*;*
g) $M \cong \mathbb{F}G_1 B^{(4)}_{\sigma,\tau}$ *and* $N \cong \mathbb{F}G_1 C^{(4)}_{s,t,u,v}$ *such that if* $s = 1$ *then* $u \neq 0$*.*

The restrictions on $C^{(4)}_{s,t,u,v}$ preventing $s = 1$ and $u = 0$ are to prevent overlap in the listing, as $C^{(4)}_{1,t,0,v} = A^{(4)}_{1,t,v}$. For the modules of the form $\mathbb{F}G_1 E^{(6)}_{r,s,t,u,v}$ only one element was chosen to represent the isomorphism class. The same was not done with the modules of Loewy length 4. □

For each of the remaining groups, we present similar diagrams and tables. In the tables listing group ring elements of $(IG)^i u$, we list the coefficients of $A^{(1)}$. In the tables listing $\|G\| \cdot m \otimes n$, we list the coefficients of $A^{(1)} \otimes A^{(1)}$.

We turn our attention to the other group which has Loewy length 7,

$$(2 \times 4).2 = \langle x, y \mid x^4 = y^4 = 1,\ {}^{x}y = y^{-1} \rangle.$$

We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1. We present the diagram for the full subgroup lattice of $(2 \times 4).2$:

Theorem 3.3. *The group ring* $\mathbb{F}(2\times 4).2$ *is decomposable into the tensor product of permutation modules,* $\mathbb{F}(2\times 4).2\|H\| \otimes \mathbb{F}(2\times 4).2\|K\|$. *Up to interchanging* H *and* K*, these are the only such pairs of subgroups:*

a) H *is one of the four subgroups* $\langle x\rangle$, $\langle xy^2\rangle$, $\langle xy\rangle$, *and* $\langle xy^{-1}\rangle$;

b) K *is one of the two subgroups* $\langle y\rangle$ *and* $\langle x^2y\rangle$.

Proof. Again, we observe the lattice diagram, and conclude that these are all the possible pairs of complementary subgroups in $(2\times 4).2$. □

We now list, up to isomorphism, all permutation modules of $\mathbb{F}(2\times 4).2$, identifying those that are uniserial on our list of all uniserial modules. To every subgroup H of $(2\times 4).2$, there is a module $\mathbb{F}(2\times 4).2\|H\|$. We list the subgroups in the order of decreasing size, with conjugate subgroups listed together. For each subgroup H, we list a generator of $\mathbb{F}(2\times 4).2\|H\|$, as described in Chapter 2, section 2.

- $\langle x, y\rangle = (2\times 4).2$ corresponds to $A^{(1)}$.
- $\langle x, y^2\rangle \cong C_2\times C_4$ corresponds to $A_0^{(2)}$.
- $\langle xy, y^2\rangle \cong C_2\times C_4$ corresponds to $A_1^{(2)}$.
- $\langle y, x^2\rangle \cong C_2\times C_4$ corresponds to $B^{(2)}$.
- $\langle x^2, y^2\rangle \cong C_2^2$ corresponds to $\bar{x}^2\bar{y}^2$, which is not a uniserial module generator.

- $C_4 \cong \langle x \rangle$ and $\langle xy^2 \rangle$ correspond to $A^{(4)}_{0,1,0}$ and $A^{(4)}_{0,0,0}$.
- $C_4 \cong \langle xy \rangle$ and $\langle xy^{-1} \rangle$ correspond to $A^{(4)}_{1,0,0}$ and $A^{(4)}_{1,1,1}$.
- $C_4 \cong \langle y \rangle$ and $\langle x^2 y \rangle$ correspond to $B^{(4)}_{0,0}$ and $B^{(4)}_{1,0}$.
- $\langle x^2 \rangle \cong C_2$ corresponds to $\bar{x}^2$, which is not a uniserial module generator.
- $\langle y^2 \rangle \cong C_2$ corresponds to $\bar{y}^2$, which is not a uniserial module generator.
- $\langle x^2 y^2 \rangle \cong C_2$ corresponds to $\bar{x}^2 + \bar{y}^2$, which is not a uniserial module generator.
- $\langle 1 \rangle$ generates all of $\mathbb{F}(2 \times 4).2$.

We now find all modules M and N such that $\mathbb{F}(2 \times 4).2 \cong M \otimes N$. We start with the case where $\dim(M) = 2$ and $\dim(N) = 8$. There is no pair of complementary subgroups H and K in $(2 \times 4).2$ such that $|H| = 8$ and $|K| = 2$. Thus there is no pair of permutation modules M and N with dimensions 2 and 8 such that $\mathbb{F}(2 \times 4).2 \cong M \otimes N$. We now look for a module N that is not a permutation module, but would suffice as an 8-dimensional factor in a tensor decomposition of $\mathbb{F}(2 \times 4).2$.

We need to find all cyclic submodules N of $\mathbb{F}(2\times 4).2$ with Poincaré polynomial $P_N = 1 + t + 2t^2 + 2t^3 + t^4 + t^5$. We use the basis from Chapter 2, Section 2 for $\mathbb{F}(2 \times 4).2$. We start with an arbitrary element of $\operatorname{Soc}^6(\mathbb{F}(2 \times 4).2)$,

$$E^{(6)} = \begin{array}{ccccccc} & & & \cdot & & & \\ & & b & & c & & \\ & d & & e & & f & \\ g & & h & & i & & j \\ & k & & l & & m & \\ & & n & & p & & \\ & & & q & & & \end{array}.$$

Let $N = \mathbb{F}(2\times 4).2E^{(6)}$. We need $E^{(6)} \notin \operatorname{Soc}^5(\mathbb{F}(2\times 4).2)$, which implies that b and c can not both be zero. Consider the map from $\operatorname{Rad}(N)$ to $\operatorname{Rad}(N)/\operatorname{Rad}^2(N)$, where we list the elements in terms of the images of the basis elements $\bar{x}^2$, $\bar{x}\bar{y}$, and $\bar{y}^2$ of $\mathbb{F}(2\times 4).2$. Using the multiplications that are given in Chapter 2, Section 2, we find that the images of $\bar{x}E^{(6)}$ and $\bar{y}E^{(6)}$ are $(b, c, 0)$ and $(0, b, b+c)$ respectively. We see that it is impossible to choose b and c such that $\dim(\operatorname{Rad}(N)/\operatorname{Rad}^2(N)) = 1$. Thus there are no $\mathbb{F}(2 \times 4).2$-modules that have this Poincaré polynomial, and no tensor decomposition $\mathbb{F}(2\times 4).2 \cong M \otimes N$ such that $\dim(M) = 2$ and $\dim(N) = 8$.

If $\mathbb{F}(2\times 4).2 \cong M \otimes N$, then both M and N must be uniserial with Loewy length 4. We give a table of all products bu, where b is one of our basis element of $\mathrm{Soc}^4(\mathbb{F}(2\times 4).2)$ that is not an element of $\mathrm{Soc}^3(\mathbb{F}(2\times 4).2)$, and u is a distinguished generator of a uniserial module of Loewy length 4. These basis elements and generators are listed in Chapter 2, section 2. We list the coefficients of $A^{(1)}$ in the products.

	$A^{(4)}_{r,s,t}$	$B^{(4)}_{s,t}$
$\bar{x}^3$	$r^2(r+1)$	1
$\bar{x}^2\bar{y}$	$r(r+1)$	0
$\bar{x}\bar{y}^2$	r	0
$\bar{y}^3$	1	0

Table of elements of $(I(2\times 4).2)^3\,u$

From this table, we determine all products,

$$\|G\|\cdot m\otimes n = \bar{x}^3 m\otimes \bar{y}^3 n + \bar{x}^2\bar{y} m\otimes \bar{x}\bar{y}^2 n + \bar{x}\bar{y}^2 m\otimes \bar{x}^2\bar{y} n + \bar{y}^3 m\otimes \bar{x}^3 n.$$

We list the coefficient of $A^{(1)}\otimes A^{(1)}$ in the product $\|G\|\cdot m\otimes n$ in the following table.

	$A^{(4)}_{r,s,t}$	$B^{(4)}_{s,t}$
$A^{(4)}_{\rho,\sigma,\tau}$	$(r+\rho)^2(1+r+\rho)$	1
$B^{(4)}_{\sigma,\tau}$	1	0

Table of coefficients of $A^{(1)}\otimes A^{(1)}$ in $\|(2\times 4).2\|\cdot m\otimes n$

From this table, we determine all tensor decompositions of $\mathbb{F}(2\times 4).2$:

Theorem 3.4. *Up to isomorphism and rearrangement of factors, the following is the complete list of non-trivial tensor decompositions,* $\mathbb{F}(2\times 4).2 \cong M\otimes N$*:*

a) $M \cong \mathbb{F}(2 \times 4).2A^{(4)}_{\rho,\sigma,\tau}$ *and* $N \cong \mathbb{F}(2 \times 4).2A^{(4)}_{r,s,t}$ *such that* $\rho + r$ *is neither 0 nor 1;*

b) $M \cong \mathbb{F}(2 \times 4).2A^{(4)}_{\rho,\sigma,\tau}$ *and* $N \cong \mathbb{F}(2 \times 4).2B^{(4)}_{s,t}$. □

We now focus on the groups G of order 16 for which $\mathbb{F}G$ has Loewy length 9: Mod_{16}, D_{16}, SD_{16}, and Q_{16}. In all four cases, the Poincaré polynomial associated to $\mathbb{F}G$ is $P_{\mathbb{F}G}(t) = 1+2t+2t^2+2t^3+2t^4+2t^5+2t^6+2t^7+t^8$. Assume $\mathbb{F}G \cong M \otimes N$. If we assume $\dim(M) = \dim(N) = 4$ then $9 \leq \ell\ell(M)+\ell\ell(N)-1 \leq 4+4-1 = 7 < 9$, which indicates a contradiction. If we assume $\dim(M) = 2$ then $9 \leq 2+\ell\ell(N)-1 \leq 1+8 = 9$, as $\ell\ell(N) \leq \dim(N)$. Thus $\ell\ell(N) = 8 = \dim(N)$, so both N and M are uniserial. The Poincaré polynomials for M and N are $P_M(t) = 1 + t$ and $P_N(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7$.

Given M and N, we wish to determine if $\mathbb{F}G \cong M \otimes N$. We only consider cases where $\dim(M)\dim(N) = 16$, $\ell\ell(M) = 2$ and $\ell\ell(N) = 8$. If we can show that the product $\|G\| \cdot M \otimes N \neq 0$, then we know that $\mathbb{F}G \cong M \otimes N$. Let m and n be generators of M and N respectively. By Lemma 1.11 of Chapter 1, the choice of generators does not matter. The product $\|G\| \cdot m \otimes n$ is non-zero if and only if the product $\|G\| \cdot M \otimes N$ is non-zero. We use Lemma 1.11 to find $\|G\| \cdot m \otimes n$.

For the groups Mod_{16} and SD_{16} we use the identity,

$$\|G\| \cdot (m \otimes n) = \bar{x}^7 m \otimes \bar{y} n + \bar{x}^6 \bar{y} m \otimes \bar{x} n,$$

where m generates a module of length 2, and n generates a module of length 8, and where $\bar{x}$ and $\bar{y}$ are defined in Sections 3 and 5 of Chapter 2.

We investigate the modular group of order 16,

$$\mathrm{Mod}_{16} = \langle x, y \mid x^8 = y^2 = 1,\ {}^{y}x = x^5 \rangle.$$

We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1. We present the diagram for the full subgroup lattice of Mod_{16}:

Theorem 3.5. *The group ring* $\mathbb{F}\mathrm{Mod}_{16}$ *is decomposable into the tensor product of permutation modules,* $\mathbb{F}\mathrm{Mod}_{16}\|H\| \otimes \mathbb{F}\mathrm{Mod}_{16}\|K\|$*. Up to interchanging* H *and* K*, these are the only such pairs of subgroups:*

a) H *is one of the two subgroups* $\langle x\rangle$ *and* $\langle xy\rangle$*;*

b) K *is one of the two subgroups* $\langle y\rangle$ *and* $\left\langle x^4 y\right\rangle$*.*

Proof. Again, we observe the lattice diagram, and conclude that these are all the possible pairs of complementary subgroups in Mod_{16}. $\square$

We now list, up to isomorphism, all permutation modules of $\mathbb{F}\mathrm{Mod}_{16}$, identifying those that are uniserial on our list of all uniserial modules. To every subgroup H of Mod_{16}, there is a module $\mathbb{F}\mathrm{Mod}_{16}\|H\|$. We list the subgroups in the order of decreasing size, with conjugate subgroups listed together. For each subgroup H, we list a generator of $\mathbb{F}\mathrm{Mod}_{16}\|H\|$, as described in Chapter 2, section 3.

- $\langle x, y\rangle = \mathrm{Mod}_{16}$ corresponds to $A^{(1)}$.
- $\langle x\rangle \cong C_8$ corresponds to $A_0^{(2)}$.
- $\langle xy\rangle \cong C_8$ corresponds to $A_1^{(2)}$.
- $\left\langle y, x^2\right\rangle \cong C_2 \times C_4$ corresponds to $B^{(2)}$.
- $\left\langle x^2\right\rangle \cong C_4$ corresponds to $\bar{x}^6$, which is not a uniserial module generator.

- $\langle x^2y\rangle \cong C_4$ corresponds to $B^{(4)}_{1,0}$.
- $\langle x^4, y\rangle \cong C_2^2$ corresponds to $B^{(4)}_{0,0}$.
- $\langle x^4\rangle \cong C_2$ corresponds to $\bar{x}^4$, which is not a uniserial module generator.
- $C_2 \cong \langle y\rangle$ and $\langle x^4y\rangle$ correspond to $B^{(8)}_{0,0,0,0}$ and $B^{(8)}_{1,0,0,0}$.
- $\langle 1\rangle$ generates all of $\mathbb{F}\mathrm{Mod}_{16}$.

We now find all modules M and N such that $\mathbb{F}\mathrm{Mod}_{16} \cong M \otimes N$. We have seen that if $\mathbb{F}\mathrm{Mod}_{16} \cong M \otimes N$ then, up to exchanging M and N, the modules must both be uniserial with dimensions 2 and 8. We give a table of all products bu, where b is a one of the basis elements and u is a distinguished generator listed in Section 3 of Chapter 3. Where the product is relevant, we list the coefficient of $A^{(1)}$ in the product.

	$B^{(8)}_{u,v,w,x}$	$A^{(2)}_r$	$B^{(2)}$
$\bar{x}^7$	1	0	0
$\bar{x}^6\bar{y}$	0	0	0
$\bar{x}$	–	r	1
$\bar{y}$	–	1	0

Table of coefficients for $A^{(1)}$ in $(I\mathrm{Mod}_{16})^n u$

From this table we determine all products,

$$\|\mathrm{Mod}_{16}\| \cdot m \otimes n = \bar{x}m \otimes \bar{x}^6\bar{y}n + \bar{y}m \otimes \bar{x}^7 n.$$

We list the coefficient of $A^{(1)} \otimes A^{(1)}$ in the product $\|\mathrm{Mod}_{16}\| \cdot m \otimes n$ in the following table.

	$A^{(2)}_r$	$B^{(2)}$
$B^{(8)}_{u,v,w,x}$	1	0

Table of coefficients of $A^{(1)} \otimes A^{(1)}$ in $\|\mathrm{Mod}_{16}\| \cdot m \otimes n$

From this table, we determine all tensor decompositions of $\mathbb{F}\mathrm{Mod}_{16}$:

Theorem 3.6. *Up to isomorphism and rearrangement of factors,* $\mathbb{F}\mathrm{Mod}_{16} \cong M \otimes N$ *where* $M \cong \mathbb{F}\mathrm{Mod}_{16} B^{(8)}_{u,v,w,x}$*, and* $N \cong \mathbb{F}\mathrm{Mod}_{16} A^{(2)}_{r}$. $\square$

We investigate the semidihedral group of order 16,

$$\mathrm{SD}_{16} = \langle x, y \mid x^8 = y^2 = 1,\ {}^{y}x = x^3 \rangle.$$

We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1. We present the diagram for the full subgroup lattice of SD_{16}:

Theorem 3.7. *The group ring* $\mathbb{F}\mathrm{SD}_{16}$ *is decomposable into the tensor product of permutation modules,* $\mathbb{F}\mathrm{SD}_{16}\|H\| \otimes \mathbb{F}\mathrm{SD}_{16}\|K\|$*. Up to interchanging* H *and* K*, these are the only such pairs of subgroups:*

a) H *is one of the two subgroups* $\langle x \rangle$ *and* $\langle x^2, xy \rangle$*;*

b) K *is one of the four conjugate subgroups* $\langle y \rangle$*,* $\langle x^4 y \rangle$*,* $\langle x^2 y \rangle$*, and* $\langle x^6 y \rangle$*.*

Proof. Again, we observe the lattice diagram, and conclude that these are all the possible pairs of complementary subgroups in SD_{16}. $\square$

We now list, up to isomorphism, all permutation modules of $\mathbb{F}\mathrm{SD}_{16}$, identifying those that are uniserial on our list of all uniserial modules. To every subgroup H of SD_{16}, there is a module $\mathbb{F}\mathrm{SD}_{16}\|H\|$. We list the subgroups in the order of decreasing size, with conjugate subgroups listed together. For each subgroup H, we list a generator of $\mathbb{F}\mathrm{SD}_{16}\|H\|$, as described in Chapter 2, section 5.

- $\langle x, y\rangle = \mathrm{SD}_{16}$ corresponds to $A^{(1)}$.
- $\left\langle xy, x^2\right\rangle \cong Q_8$ corresponds to $A_1^{(2)}$.
- $\langle x\rangle \cong C_8$ corresponds to $A_0^{(2)}$.
- $\left\langle y, x^2\right\rangle \cong D_8$ corresponds to $B^{(2)}$.
- $\left\langle x^2\right\rangle \cong C_4$ corresponds to $\bar{x}^6$, which is not a uniserial module generator.
- $C_4 \cong \langle xy\rangle$ and $\left\langle x^3 y\right\rangle$ correspond to $A_{1,0}^{(4)}$ and $A_{0,0}^{(4)}$.
- $C_2^2 \cong \left\langle y, x^4\right\rangle$ and $\left\langle x^2 y, x^4\right\rangle$ correspond to $B_{0,0}^{(4)}$ and $B_{1,0}^{(4)}$.
- $\left\langle x^4\right\rangle \cong C_2$ corresponds to $\bar{x}^4$, which is not a uniserial module generator.
- $C_2 \cong \langle y\rangle$, $\left\langle x^4 y\right\rangle$, $\left\langle x^2 y\right\rangle$, and $\left\langle x^6 y\right\rangle$ correspond to $B_{0,0,0,0}^{(8)}$, $B_{0,1,0,0}^{(8)}$, $B_{1,1,1,0}^{(8)}$ and $B_{1,0,0,0}^{(8)}$.
- $\langle 1\rangle$ generates all of $\mathbb{F}\mathrm{SD}_{16}$.

We now find all modules M and N such that $\mathbb{F}\mathrm{SD}_{16} \cong M \otimes N$. We have seen that if $\mathbb{F}\mathrm{SD}_{16} \cong M \otimes N$ then, up to exchanging M and N, the modules must both be uniserial with dimensions 2 and 8. We give a table of all products bu, where b is a one of the basis elements and u is a distinguished generator listed in Section 3 of Chapter 3. Where the product is relevant, we list the coefficient of $A^{(1)}$ in the product.

	$B_{s,u,w,x}^{(8)}$	$A_r^{(2)}$	$B^{(2)}$
$\bar{x}$	–	r	1
$\bar{y}$	–	1	0
$\bar{x}^7$	1	0	0
$\bar{x}^6\bar{y}$	0	0	0

Table of coefficients for $A^{(1)}$ in $(I\mathrm{SD}_{16})^n u$

From this table we determine all products,

$$\|\mathrm{SD}_{16}\| \cdot m \otimes n = \bar{x}m \otimes \bar{x}^6\bar{y}n + \bar{y}m \otimes \bar{x}^7 n.$$

We list the coefficient of $A^{(1)} \otimes A^{(1)}$ in the product $\|\mathrm{SD}_{16}\| \cdot m \otimes n$ in the following table.

	$A_r^{(2)}$	$B^{(2)}$
$B_{s,u,w,x}^{(8)}$	1	0

Table of coefficients of $A^{(1)} \otimes A^{(1)}$ in $\|\mathrm{SD}_{16}\| \cdot m \otimes$n

From this table, we determine all tensor decompositions of $\mathbb{F}\mathrm{SD}_{16}$:

Theorem 3.8. *Up to isomorphism and rearrangement of factors,* $\mathbb{F}\mathrm{SD}_{16} \cong M \otimes N$ *where* $M \cong \mathbb{F}\mathrm{SD}_{16} B_{u,v,w,x}^{(8)}$*, and* $N \cong \mathbb{F}\mathrm{SD}_{16} A_r^{(2)}$. □

We investigate the quaternion group of order 16,

$$\mathrm{Q}_{16} = \langle x, y \mid x^8 = 1, x^4 = y^2,\ {}^{y}x = x^{-1} \rangle.$$

We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1. We present the diagram for the full subgroup lattice of Q_{16} after the theorem.

Theorem 3.9. *The group ring* $\mathbb{F}\mathrm{Q}_{16}$ *is not decomposable into the tensor product of permutation modules,* $\mathbb{F}\mathrm{Q}_{16}\|H\| \otimes \mathbb{F}\mathrm{Q}_{16}\|K\|$.

Proof. We observe the lattice diagram, and conclude that all non-trivial subgroups of Q_{16} intersect in the subgroup $\langle x^4 \rangle$. Thus there is no pair of non-trivial complementary subgroups in Q_{16}. □

We now list, up to isomorphism, all permutation modules of $\mathbb{F}\mathrm{Q}_{16}$, identifying those that are uniserial on our list of all uniserial modules. To every subgroup H of Q_{16}, there is a module $\mathbb{F}\mathrm{Q}_{16}\|H\|$. We list the subgroups in the order of decreasing size, with conjugate subgroups listed together. For each subgroup H, we list a generator of $\mathbb{F}\mathrm{Q}_{16}\|H\|$, as described in Chapter 2, section 6.

- $\langle x, y\rangle = \mathrm{Q}_{16}$ corresponds to $A^{(1)}$.
- $\langle x\rangle \cong C_8$ corresponds to $A_0^{(2)}$.
- $\left\langle y, x^2\right\rangle \cong Q_8$ corresponds to $B^{(2)}$.
- $\left\langle xy, x^2\right\rangle \cong Q_8$ corresponds to $A_1^{(2)}$.
- $\left\langle x^2\right\rangle \cong C_4$ corresponds to $\bar{x}^6$, which is not a uniserial module generator.
- $C_4 \cong \langle y\rangle$ and $\left\langle x^2 y\right\rangle$ correspond to $B_{0,0}^{(4)}$ and $B_{1,0}^{(4)}$.
- $C_4 \cong \langle xy\rangle$ and $\left\langle x^3 y\right\rangle$ correspond to $A_{1,0}^{(4)}$ and $A_{0,0}^{(4)}$.
- $\left\langle x^4\right\rangle \cong C_2$ corresponds to $\bar{x}^4$, which is not a uniserial module generator.
- $\langle 1\rangle$ generates all of $\mathbb{F}\mathrm{Q}_{16}$.

Theorem 3.10. *There are no non-trivial tensor decompositions of* $\mathbb{F}\mathrm{Q}_{16}$.

Proof. From Chapter 2, Section 6, we see that $\mathbb{F}\mathrm{Q}_{16}$ has no uniserial modules of length 8. Therefore, there is no non-trivial tensor decomposition of $\mathbb{F}\mathrm{Q}_{16}$. □

We investigate the dihedral group of order 16,

$$\mathrm{D}_{16} = \left\langle x, y, z \mid x^2 = y^2 = (xy)^8, z = xy \right\rangle .$$

We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1. We present the diagram for the full subgroup lattice of D_{16}:

Theorem 3.11. *The group ring $\mathbb{F}\mathrm{D}_{16}$ is decomposable into the tensor product of permutation modules, $\mathbb{F}\mathrm{D}_{16}\|H\| \otimes \mathbb{F}\mathrm{D}_{16}\|K\|$. Up to interchanging H and K, these are the only such pairs of subgroups:*

a) H is the subgroup $\left\langle x, z^2 \right\rangle$ and K is one of the four subgroups $\langle y \rangle$, $\left\langle z^3 x \right\rangle$, $\langle zx \rangle$, or $\left\langle yz^2 \right\rangle$;

b) H is the subgroup $\left\langle y, z^2 \right\rangle$ and K is one of the four subgroups $\langle x \rangle$, $\left\langle yz^3 \right\rangle$, $\langle yz \rangle$, or $\left\langle z^2 x \right\rangle$;

c) H is the subgroup $\langle z \rangle$ and K is any of the subgroups of order two other than $\left\langle z^4 \right\rangle$.

Proof. Again, we observe the lattice diagram, and conclude that these are all the possible pairs of complementary subgroups in D_{16}. Note that there is an automorphism of D_{16} which converts the pairs of statement a) into the pairs of statement b). $\square$

We now list, up to isomorphism, all permutation modules of $\mathbb{F}\mathrm{D}_{16}$, identifying those that are uniserial on our list of all uniserial modules. We list all subgroups H of D_{16} along with a generator of the module generated by $\|H\|$. Subgroups are listed in decreasing order, and they are listed by conjugacy class.

- $\langle x, y\rangle = \mathrm{D}_{16}$ corresponds with $A^{(1)}$
- $\left\langle x, z^2\right\rangle \cong \mathrm{D}_8$ corresponds with $A_0^{(2)}$
- $\left\langle y, z^2\right\rangle \cong \mathrm{D}_8$ corresponds with $B^{(2)}$
- $\langle z\rangle \cong \mathrm{C}_8$ corresponds with $A_1^{(2)}$
- $\mathrm{C}_2 \times \mathrm{C}_2 \cong \left\langle x, z^4\right\rangle$ and $\left\langle yz, z^4\right\rangle$ correspond with $A_{0,0}^{(4)}$ and $A_{1,0}^{(4)}$ respectively
- $\mathrm{C}_2 \times \mathrm{C}_2 \cong \left\langle y, z^4\right\rangle$ and $\left\langle zx, z^4\right\rangle$ correspond with $B_{0,0}^{(4)}$ and $B_{1,0}^{(4)}$ respectively
- $\left\langle z^2\right\rangle \cong \mathrm{C}_4$ corresponds with $A_0^{(3)} + B_0^{(3)}$, which generates a 4-dimensional module
- $\mathrm{C}_2 \cong \langle x\rangle$, $\left\langle yz^3\right\rangle$, $\langle yz\rangle$, and $\left\langle z^2x\right\rangle$ correspond with $A_{0,0,0}^{(8)}$, $A_{0,1,1}^{(8)}$, $A_{1,0,0}^{(8)}$ and $A_{1,1,1}^{(8)}$ respectively
- $\mathrm{C}_2 \cong \langle y\rangle$, $\left\langle z^3x\right\rangle$, $\langle zx\rangle$, and $\left\langle yz^2\right\rangle$ correspond with $B_{0,0,0}^{(8)}$, $B_{0,1,1}^{(8)}$, $B_{1,0,0}^{(8)}$ and $B_{1,1,1}^{(8)}$ respectively
- $\mathrm{C}_2 \cong \left\langle z^4\right\rangle$ corresponds with $A_0^{(5)} + B_0^{(5)}$, which generates an 8-dimensional module
- $\langle 1\rangle$ generates the entire module $\mathbb{F}\mathrm{D}_{16}$.

We now find all modules M and N such that $\mathbb{F}\mathrm{D}_{16} \cong M \otimes N$. Up to exchanging M and N, the modules must both be uniserial with dimensions 2 and 8. We give a table of all products bu, where b is one of the basis elements and u is a distinguished generator listed in section 4 of chapter 2. Where the product is relevant, we list the coefficient of $A^{(1)}$ in the product.

	$A^{(8)}_{s,u,w,x}$	$B^{(8)}_{s,u,w,x}$	$A^{(2)}_r$	$B^{(2)}$
$\bar{x}(\bar{y}\bar{x})^3$	0	1	0	0
$\bar{y}(\bar{x}\bar{y})^3$	1	0	0	0
$\bar{x}$	–	–	r	1
$\bar{y}$	–	–	1	0

Table of coefficients for $A^{(1)}$ in $(I\mathrm{D}_{16})^n u$

From this table we determine all products,

$$\|D_{16}\| \cdot m \otimes n = \bar{x}m \otimes \bar{y}(\bar{x}\bar{y})^3 n + \bar{y}m \otimes \bar{x}(\bar{y}\bar{x})^3 n.$$

We list the coefficients of $A^{(1)} \otimes A^{(1)}$ in the product $\|D_{16}\| \cdot m \otimes n$ in the following table.

	$A^{(2)}_r$	$B^{(2)}$
$A^{(8)}_{s,u,w,x}$	r	1
$B^{(8)}_{s,u,w,x}$	1	0

Table of coefficients of $A^{(1)} \otimes A^{(1)}$ in $\|\mathrm{D}_{16}\| \cdot m \otimes n$

From this table, we determine all tensor decompositions of $\mathbb{F}\mathrm{D}_{16}$:

Theorem 3.12. *Up to isomorphism and rearrangement of factors, the following is the complete list of non-trivial tensor decompositions,* $\mathbb{F}\mathrm{D}_{16} \cong M \otimes N$*:*

a) $M \cong \mathbb{F}\mathrm{D}_{16}A^{(2)}_r$ *and* $N \cong \mathbb{F}\mathrm{D}_{16}A^{(8)}_{s,u,w,x}$ *when* $\rho \neq 0$*;*
b) $M \cong \mathbb{F}\mathrm{D}_{16}A^{(2)}_r$ *and* $N \cong \mathbb{F}\mathrm{D}_{16}B^{(8)}_{s,u,w,x}$
c) $M \cong \mathbb{F}\mathrm{D}_{16}B^{(2)}$ *and* $N \cong \mathbb{F}\mathrm{D}_{16}A^{(8)}_{s,u,w,x}$

The remaining group G is such that the Loewy length of $\mathbb{F}G$ is 6:

$$\mathrm{D}_8\mathrm{Y}4 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, x \in Z, (yz)^2 = x^2 \rangle.$$

The expression $x \in Z$ indicates that x is in the center of $\mathrm{D}_8\mathrm{Y4}$. This group is the central product of D_8 and C_4.

The Poincaré polynomial associated with $\mathbb{F}\mathrm{D}_8\mathrm{Y4}$ is $P_{\mathbb{F}\mathrm{D}_8\mathrm{Y4}}(t) = 1+3t+4t^2+4t^3+3t^4+t^5$. Assume $\mathbb{F}\mathrm{D}_8\mathrm{Y4} \cong M \otimes N$. If we assume $\dim(M) = \dim(N) = 4$ then $6 \leq \ell\ell(M)+\ell\ell(N)-1 \leq 4+4-1 = 7$. At least one of the modules must have Loewy length 4, and is thus uniserial. But from Section 7 of Chapter 2 we see that there are no uniserial modules of length 4, so this is not a possibility. If we assume $\dim(M) = 2$ then $6 \leq 2+\ell\ell(N)-1 \leq 1+5 = 6$, as $\ell\ell(N) < \ell\ell(\mathbb{F}\mathrm{D}_8\mathrm{Y4})$. Thus we must have $\ell\ell(N) = 5$. The Poincaré polynomials for M and N are $P_M(t) = 1+t$ and $P_N(t) = 1+2t+2t^2+2t^3+t^4$.

Given M and N, we wish to determine if $\mathbb{F}G \cong M \otimes N$. We only consider cases where $\dim(M)\dim(N) = 16$, $\ell\ell(M) = 2$ and $\ell\ell(N) = 5$. If we can show that the product $\|G\| \cdot M \otimes N \neq 0$, then we know that $\mathbb{F}G \cong M \otimes N$. Let m and n be generators of M and N respectively. By Lemma 1.11 of Chapter 1, the choice of generators does not matter. The product $\|G\| \cdot m \otimes n$ is non-zero if and only if the product $\|G\| \cdot M \otimes N$ is non-zero. We use Lemma 1.11 to find $\|G\| \cdot m \otimes n$. We find that

$$\|G\| \cdot (m \otimes n) = \bar{x}m \otimes \bar{x}^2\bar{y}\bar{z}n + \bar{y}m \otimes \bar{x}^3\bar{z}n + \bar{z}m \otimes \bar{x}^3\bar{y}n,$$

where m generates a module of length 2, and n generates a module of length 8, and where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are defined in Section 7 of Chapter 2.

We first consider the decomposition of this group into the non-trivial product of subgroups H and K whose intersection is 1. We present the diagram for the full subgroup lattice of $\mathrm{D}_8\mathrm{Y4}$ after the theorem.

Theorem 3.13. *The group ring* $\mathbb{F}\mathrm{D}_8\mathrm{Y4}$ *is decomposable into the tensor product of permutation modules,* $\mathbb{F}\mathrm{D}_8\mathrm{Y4}\|H\| \otimes \mathbb{F}\mathrm{D}_8\mathrm{Y4}\|K\|$. *Up to interchanging* H *and* K*, these are the only such pairs of subgroups:*

a) H *is one of the four subgroups* $\langle xy, xz\rangle$, $\langle x, y\rangle$, $\langle x, yz\rangle$, *or* $\langle xz, y\rangle$ *and* K *is one of the conjugate subgroups* $\langle z\rangle$ *or* $\langle x^2z\rangle$;

b) H *is one of the four subgroups* $\langle xy, xz\rangle$, $\langle x, z\rangle$, $\langle x, yz\rangle$, *or* $\langle xy, z\rangle$ *and* K *is one of the conjugate subgroups* $\langle y\rangle$ *or* $\langle x^2y\rangle$;

c) H is one of the four subgroups $\langle xy, xz\rangle$, $\langle x, y\rangle$, $\langle x, z\rangle$, or $\langle y, z\rangle$ and K is one of the conjugate subgroups $\langle xyz\rangle$ or $\left\langle x^{-1}yz\right\rangle$;

Proof. Again, we observe the lattice diagram, and conclude that these are all the possible pairs of complementary subgroups in $\mathrm{D}_8\mathrm{Y}4$. Note that for each pair of statements of the theorem, there is an isomorphism on $\mathrm{D}_8\mathrm{Y}4$ which converts one of the statements to the other. $\square$

We now list, up to isomorphism, all permutation modules of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$, identifying those that are uniserial on our list of all uniserial modules. We also identify 8-dimensional permutation modules that have Loewy length 5 with a generator described later. To every subgroup H of $\mathrm{D}_8\mathrm{Y}4$, there is a module $\mathbb{F}\mathrm{D}_8\mathrm{Y}4\|H\|$. We list the subgroups in the order of decreasing size, with conjugate subgroups listed together. For each subgroup H, we list a generator of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4\|H\|$, as described in Chapter 2, section 7.

- $\langle x, y, z\rangle = \mathrm{D}_8\mathrm{Y}4$ corresponds to $A^{(1)}$.
- $\langle xy, xz\rangle \cong Q_8$ corresponds to $A^{(2)}_{1,1}$.
- $\langle x, y\rangle \cong C_2 \times C_4$ corresponds to $B^{(2)}_0$.
- $\langle x, z\rangle \cong C_2 \times C_4$ corresponds to $C^{(2)}$.

- $\langle x, yz\rangle \cong C_2 \times C_4$ corresponds to $B_1^{(2)}$.
- $\langle z, xy\rangle \cong D_8$ corresponds to $A_{0,1}^{(2)}$.
- $\langle y, xz\rangle \cong D_8$ corresponds to $A_{1,0}^{(2)}$.
- $\langle y, z\rangle \cong D_8$ corresponds to $A_{0,0}^{(2)}$.
- $\langle x\rangle \cong C_4$ corresponds to $\bar{x}^3$, which is not a uniserial module generator.
- $\langle xy\rangle \cong C_4$ corresponds to $\bar{x}^3+\bar{x}^2\bar{y}$, which is not a uniserial module generator.
- $\langle xz\rangle \cong C_4$ corresponds to $\bar{x}^3+\bar{x}^2\bar{z}$, which is not a uniserial module generator.
- $\langle yz\rangle \cong C_4$ corresponds to $\bar{x}^2(\bar{y}+\bar{z})$, which is not a uniserial module generator.
- $\left\langle x^2, y\right\rangle \cong C_2^2$ corresponds to $\bar{x}^2\bar{y}$, which is not a uniserial module generator.
- $\left\langle x^2, z\right\rangle \cong C_2^2$ corresponds to $\bar{x}^2\bar{z}$, which is not a uniserial module generator.
- $\left\langle x^2, xyz\right\rangle \cong C_2^2$ corresponds to $\bar{x}^3+\bar{x}^2(\bar{y}+\bar{z}+\bar{y}\bar{z})$, which is not a uniserial module generator.
- $\left\langle x^2\right\rangle \cong C_2$ corresponds to $\bar{x}^2$, which is not a uniserial module generator.
- $C_2 \cong \langle y\rangle$ and $\left\langle x^2 y\right\rangle$ correspond to $E_{0,0,0,0}^{(5)}$ and $E_{1,0,0,0}^{(5)}$.
- $C_2 \cong \langle z\rangle$ and $\left\langle x^2 z\right\rangle$ correspond to $F_{0,0,0,0}^{(5)}$ and $F_{1,0,0,0}^{(5)}$.
- $C_2 \cong \langle xyz\rangle$ and $\left\langle x^{-1}yz\right\rangle$ correspond to $D_{1,0,0,0}^{(5)}$ and $D_{0,0,0,0}^{(5)}$.
- $\langle 1\rangle$ generates all of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$. We now find all modules M and N such that $\mathbb{F}\mathrm{D}_8\mathrm{Y}4 \cong M \otimes N$. We have seen that if $\mathbb{F}\mathrm{D}_8\mathrm{Y}4 \cong M \otimes N$ then, up to exchanging M and N, the modules have dimensions 2 and 8, respectively, and N must have Loewy length 5, and $P_N = 1+2t+2t^2+2t^3+t^4$. We now find all cyclic submodules N of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$ with Poincaré polynomial $P_N = 1+t+2t^2+2t^3+t^4+t^5$. We use an algorithm similar to that described when we were dealing with G_1. We sketch the use of the algorithm, although the actual calculations are performed by computer, and are not shown here. We use the basis from Chapter 2, Section 7 for $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$. We start with an arbitrary element of $\mathrm{Soc}^6(\mathbb{F}\mathrm{D}_8\mathrm{Y}4)$,

$$D^{(5)} = \begin{array}{cccccc} & a & & & & \\ b & & c & & d & \\ e & & f & g & & h \\ i & & j & k & & l \\ & m & & n & & p \\ & & & & q & \end{array}.$$

Let $N = \mathbb{F}\mathrm{D}_8\mathrm{Y}4D^{(5)}$. We need $D^{(5)} \notin \mathrm{Soc}^4(\mathbb{F}\mathrm{D}_8\mathrm{Y}4)$. This implies that b, c and d can not all be zero. Consider the map from $\mathrm{Rad}(N)$ to $\mathrm{Rad}(N)/\mathrm{Rad}^2(N)$, where we list the elements in terms of the images of the basis elements $\bar{x}^2$, $\bar{x}\bar{y}$, $\bar{x}\bar{z}$, and $\bar{y}\bar{z}$ of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$. Using the multiplications that are given in Chapter 2, Section 7, we find that the images of $\bar{x}D^{(5)}$, $\bar{y}D^{(5)}$ and $\bar{z}D^{(5)}$ are $(b, c, d, 0)$, $(0, b, 0, d)$ and $(c, 0, b, c)$ respectively. We may assume that the first non-zero element in the list $\{b, c, d\}$ is equal to 1, as we may rescale. If $b = 1$, we must have $cd = 1$. If $b = 0$ and $c = 1$, then we must have $d = 0$. If $b = c = 0$, the case where $d = 1$ is also valid. Further quotients of powers of the radical show that all three of these outcomes are possible, with the added restriction that if $b = 1$ then c and d equal 1 as well.

For each of these three cases, we find the set of elements $\bar{x}^i\bar{y}^j\bar{z}^kE^{(6)}$, and perform a Gram-Schmidt type of process on the set in the same manner we described with G_1 . In each of the three cases, we may assume that some of the entries are zero. In each case, there is also a restriction on the possible values of the remaining variables. We list the three possible families of distinguished module generators here, and rename them:

$$D^{(5)}_{r,s,t,u} = \begin{array}{cccccc} & \cdot & & & & \\ 1 & & 1 & & 1 & \\ \cdot & & \cdot & r & & 1+r \\ \cdot & & \cdot & s & & t \\ & \cdot & & \cdot & & u \\ & & & & \cdot & \end{array}, \quad E^{(5)}_{r,s,t,u} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & 1 & & \cdot & \\ \cdot & & \cdot & \cdot & & r \\ \cdot & & \cdot & s & & t \\ & \cdot & & u & & \cdot \\ & & & & \cdot & \end{array},$$

$$F^{(5)}_{r,s,t,u} = \begin{array}{cccccc} & \cdot & & & & \\ \cdot & & \cdot & & 1 & \\ r & & \cdot & \cdot & & \cdot \\ s & & t & \cdot & & \cdot \\ & u & & \cdot & & \cdot \\ & & & & \cdot & \end{array}$$

We give a table of all products bu, where b is a one of the basis elements and u is a distinguished generator listed in Section 3 of Chapter 7, or one of those just described. We list the coefficient of $A^{(1)}$ in the product.

	$D^{(5)}_{r,s,t,u}$	$E^{(5)}_{r,s,t,u}$	$F^{(5)}_{r,s,t,u}$
$\bar{x}^2\bar{y}\bar{z}$	1	0	0
$\bar{x}^3\bar{z}$	1	1	0
$\bar{x}^3\bar{y}$	1	0	1

Table of coefficients for $A^{(1)}$ in $(I\mathrm{D}_8\mathrm{Y}4)^4u$

	$A^{(2)}_{r,s}$	$B^{(2)}_r$	$C^{(2)}$
$\bar{x}$	1	0	0
$\bar{y}$	s	r	1
$\bar{z}$	r	1	0

Table of coefficients for $A^{(1)}$ in $(I\mathrm{D}_8\mathrm{Y}4)u$

From this table we determine all products,

$$\|\mathrm{D}_8\mathrm{Y}4\| \cdot m \otimes n = \bar{x}m \otimes \bar{x}^2\bar{y}\bar{z}n + \bar{y}m \otimes \bar{x}^3\bar{z}n + \bar{z}m \otimes \bar{x}^3\bar{y}n.$$

We list the coefficient of $A^{(1)} \otimes A^{(1)}$ in the product $\|\mathrm{D}_8\mathrm{Y}4\| \cdot m \otimes n$ in the following table.

	$A^{(2)}_{\rho,\sigma}$	$B^{(2)}_\rho$	$C^{(2)}$
$D^{(5)}_{r,s,t,u}$	$1+\rho+\sigma$	$1+\rho$	1
$E^{(5)}_{r,s,t,u}$	σ	ρ	1
$F^{(5)}_{r,s,t,u}$	ρ	1	0

Table of coefficients of $A^{(1)} \otimes A^{(1)}$ in $\|\mathrm{D}_8\mathrm{Y}4\| \cdot m \otimes n$

From this table, we determine all tensor decompositions of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$:

Theorem 3.14. *Up to isomorphism and rearrangement of factors, the following is the complete list of non-trivial tensor decompositions,* $\mathbb{F}\mathrm{D}_8\mathrm{Y}4 \cong M \otimes N$*:*

a) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4A^{(2)}_{\rho,\sigma}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4D^{(5)}_{r,s,t,u}$ *when* $\rho + \sigma \neq 1$*;*
b) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4A^{(2)}_{\rho,\sigma}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4E^{(5)}_{r,s,t,u}$ *when* $\sigma \neq 0$*;*
c) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4A^{(2)}_{\rho,\sigma}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4F^{(5)}_{r,s,t,u}$ *when* $\rho \neq 0$*;*
d) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4B^{(2)}_{\rho}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4D^{(5)}_{r,s,t,u}$ *when* $\rho \neq 1$*;*
e) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4B^{(2)}_{\rho}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4E^{(5)}_{r,s,t,u}$ *when* $\rho \neq 0$*;*
f) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4B^{(2)}_{\rho}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4F^{(5)}_{r,s,t,u}$*;*
g) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4C^{(2)}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4D^{(5)}_{r,s,t,u}$*;*
h) $M \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4C^{(2)}$ *and* $N \cong \mathbb{F}\mathrm{D}_8\mathrm{Y}4E^{(5)}_{r,s,t,u}$*.*

We may further factor $P_N(t) = (1+t)(1+t+t^2+t^3)$, so it seems possible that $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$ may be factored into the tensor product of 3 modules, $\mathbb{F}\mathrm{D}_8\mathrm{Y}4 \cong L \otimes M \otimes N$. We could rearrange the orders of L, M, and N so that $\dim(L) = \dim(M) = 2$ and $\dim(N) = 4$. However, we know that $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$ has no decomposition into the product of two 4-dimensional modules. This implies that the modules $L \otimes M$ and N could not exist, so there is no non-trivial factorization of $\mathbb{F}\mathrm{D}_8\mathrm{Y}4$ into a triple tensor product.

Bibliography

[Al 1] J. L. Alperin, 'Loewy Structure of Permutation Modules for p-Groups', *Quart. J. Math. Oxford* (2), 39 (1988), 129–133.

[Al 2] J. L. Alperin, *Local Representation Theory*, Cambridge University Press, Cambridge, 1986.

[As] Michael Aschbacher, *Finite Group Theory*, Cambridge University Press, Cambridge, 1986.

[AR&S] M. Auslander, I. Reiten, and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge University Press, Cambridge, 1995.

[B&C] D. J. Benson and J. F. Carlson, 'Complexity and Multiple Complexes', *Math. Z.* 195 (1987), 221–238.

[B] K. Bongartz, 'A Note on Algebras of Finite Uniserial Type', *J. Algebra* 188 (1997), 513–515.

[C&K] Jon F. Carlson and L. G. Kovács, 'Tensor Factorization of Group Algebras and Modules', *Journal of Algebra* 175 (1995), 385–407.

[F] Kent R. Fuller, 'Biserial rings', in *Ring Theory* 64–90 in Springer Lecture Notes in Math 734 (1979).

[H&S] Marshall Hall, Jr. and James K. Senior, *The Groups of Order* 2^n $(n \leq 6)$, The MacMillan Company, New York, 1964.

[H] Thomas W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.

[J] S. Jennings, 'The Structure of the Group Ring of a p-Group over a Module Field', *Trans. Amer. Math. Soc.* 50 (1941), 175–185.

[LG] C. R. Leedham-Green, 'Structure of Finite p-groups', J. London Math. Soc. 50 (1994), 49–67.

[O] Philip Osterlund 'A Diagrammatic Approach to Projective Resolutions of Modules Over Group Rings', unpublished.

[P] G. Pazderski, 'Representations of p-groups at Characteristic p I', Beitrage zur Algebra und Geometrie 10 (1980), 97–104, M.R. 83i:20014.

[R] Joseph J. Rotman *Theory of Groups*, Wm. C. Brown Publishers, Dubuque, 1988.

[Sh] Aner Shalev, 'Uniserial Permutation Modules', Quarterly J. Math 42 (1991), 375–378.

[Sc] Martin Schönert *et al* (1994), GAP — *Groups Algorithms and Programming.* Lehrstuhl D für Mathematik, RWTH, Aachen.

[ZH] B. Zimmermann-Huisgen, 'The Geometry of Uniserial Representations of Finite-Dimensional Algebras III: Finite Uniserial Type', *Trans. Amer. Math. Soc. 348* (1996), 4775–4812.

Philip Lloyd Osterlund

178 words

Abstract

This thesis is aimed towards the determination of tensor decompositions of the regular representation of finite p-groups over fields of characteristic p. We also focus on uniserial modules, as they are often factors. We develop an algorithm that determines, up to isomorphism, all uniserial modules of a group ring (Section 2, pp. 11–12). In a tensor decomposition of the regular representation of a group, we relate the Loewy lengths, radical series and socle series of the regular representation to those of the factors (Theorems 1.4–1.7, pp. 69–74). Under certain conditions, we find that the Poincaré polynomial associated to the filtration of a module is the product of the corresponding Poincaré polynomials of the module's tensor factors (Corollary 1.8, p. 73). For the non-abelian groups of order 16 that can not be written as direct products of proper subgroups we classify the uniserial modules (Chapter 2, pp. 23–66) and all tensor decompositions (Section 3, pp. 85–111).

Key words Tensor decomposition, uniserial module, p-group, radical and socle series, Poincaré polynomial.

Contents

Figures

Tables