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GRADUATE SCHOOL

Equivariant Homotopy Type of Categories and Preordered Sets

A thesis
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Contents

Chapter 1

Introduction

Let G be a finite group. In several instances it has proven to be fruitful to have a space on which G acts in order to study the group G (consider [Qui78], [AM94], [AM92] [AMM91], [DD89], [Ser80] and [Bro89]). For example, if $\mathrm{B}G$ is the classifying space of G, characterized by being a connected CW-complex with fundamental group G and higher homotopy groups zero, then G acts freely on its universal cover $\mathrm{E}G$, which is a contractible space. The homology and the cohomology of G can be defined as the homology or cohomology of the space $\mathrm{B}G$.

Many of the formulas that have been used to study group cohomology are obtained from a simplicial complex Δ on which G acts. The simplicial complex often comes from a partially ordered set (poset) where the points are subgroups of G, the order relation is containment, and the action of G is given by conjugation. The complexes of Brown ([Bro75]), Quillen ([Qui78]) and Bouc ([Web87b]) are examples of this.

From an action of a finite group G on a simplicial complex Δ satisfying some suitable properties, Webb obtains in [Web91] an acyclic split chain complex involving the p-part of the cohomology of G (where p is a prime number) and that of the stabilizers of the simplices of Δ. In [Dwy97] and [Dwy98b], W. Dwyer considers some new spaces which are canonically constructed from the combinatorial structure of the subgroups of G but which do not have the same equivariant homotopy type (in general) as the complexes of Brown, Quillen and Bouc, although they do have the same ordinary homotopy type. These spaces are the nerves of certain categories defined in terms of subgroups of G. In these spaces the stabilizers are the subgroups themselves in one case, or their centralizers in the other.

In this thesis I give theorems, some of them new, that generalize well known theorems about equivariant homotopy type of posets (see for example [TW91] and [Wel95]) and about homotopy type of simplicial sets. Given a category on which G acts, they allow us to obtain a simpler category with the same G-homotopy type, and so preserving the necessary properties to apply Webb's theorems.

In Chapter 2, we define the objects we will study: categories, simplicial and bisimplicial sets, and topological spaces with a group action. We will also explore basic relationships among them. In Chapter 3 we state the theorems that hold in the general context of categories with a group action. We introduce the new concept of an action of G by natural transformations on a functor $F\colon \mathbf{C} \to \mathbf{D}$, which allows us to define a G-action on $\operatorname{hocolim} F$ in the case that $\mathbf{C}$ is a G-category, but not necessarily $\mathbf{D}$ is. We prove the most important property of hocolim in this new context: the equivariant homotopy invariance (Theorem 3.20). We finish the chapter with a proof of the equivariant version of a theorem of Thomason, which expresses any homotopy colimit of a functor to small categories as the nerve of a certain category. In Chapter 4 we write what the theorems of the previous chapter mean in the more specialized setting of preordered sets with a group action, give some definitions and prove some further theorems. We prove that the nerve of any preordered set with a group action can be expressed in an equivariant fashion as the homotopy colimit of a functor defined on the associated poset with an action by natural transformations and whose values are ordinarily contractible. In Chapter 5 we consider a particular G-preordered set, which is due to Dwyer, defined in terms of a family of subgroups of G. We prove theorems that help us simplify such a family of subgroups without losing any essential information. They are also applied to obtain results analogous to those of Quillen on the join of posets and on connected components. In Chapter 6 we study the chain complex of a G-preordered set, which is a chain complex of G-modules. Finally we obtain as applications of this theory a new exact sequence that relates the cohomology of a group to that of its subgroups and a spectral sequence that contains the mentioned exact sequence as a particular case.

Chapter 2

Basic Definitions

In this chapter we review some basic ideas to do with categories and their nerves, and we say what we mean by the equivariant versions of these notions.

2.1 Categories with a Group Action

For the basic material about categories, we refer the reader to [Mac71]. Throughout, G will be a fixed finite group, and for a given category $\mathbf{D}$ we start by defining what we mean by the category of G-objects in $\mathbf{D}$. The most convenient way to do this is to regard G as a category $\mathbf{G}$ with a single object $*$, in which $\hom_{\mathbf{G}}(*, *) = G$ and the composition is equal to the group multiplication. Now, the category of G-objects in $\mathbf{D}$ is the category $\mathbf{D}^{\mathbf{G}}$ of functors from $\mathbf{G}$ to $\mathbf{D}$. Note that if G is the trivial group, then $\mathbf{D}^{\mathbf{G}}$ is just $\mathbf{D}$.

For example, if $\mathbf{D} = \mathbf{SCat}$, the category of small categories, each functor $F\colon \mathbf{G} \to \mathbf{SCat}$ is determined by the small category $F(*) = \mathbf{C}$, and actions of G on both the set $\operatorname{obj} \mathbf{C}$ and the set $\cup_{A,B \in \operatorname{obj} \mathbf{C}} \hom_{\mathbf{C}}(A, B)$ in such a way that

1. $\phi \in \hom_{\mathbf{C}}(A, B)$ implies $g\phi \in \hom_{\mathbf{C}}(gA, gB)$ for all $g \in G$,

2. $g1_A = 1_{gA}$ for all $g \in G$ and $A \in \operatorname{obj} \mathbf{C}$,

3. $g(\phi \circ \psi) = (g\phi) \circ (g\psi)$ for all $g \in G$, and ϕ, ψ maps in $\mathbf{C}$.

We say then that $\mathbf{C}$ is a *G-category*. If $\mathbf{C}_1$ and $\mathbf{C}_2$ are G-categories, determined respectively by functors $F_1, F_2\colon \mathbf{G} \to \mathbf{SCat}$, then a *$G$-functor* $\mathbf{C}_1 \to \mathbf{C}_2$ is a natural transformation from F_1 to F_2. Hence a G-functor is determined by a functor $\eta\colon \mathbf{C}_1 \to \mathbf{C}_2$ such that $\eta(gA) = g\eta(A)$ for all objects A in $\mathbf{C}$ and $g \in G$, and $\eta(g\phi) = g\eta(\phi)$ for all morphisms ϕ in $\mathbf{C}$ and $g \in G$. A G-functor is also called an *equivariant functor*.

2.2 Simplicial Objects with a Group Action

For the basic notions about simplicial objects we will refer the reader to [May67] and [GJ97]. We use $\mathbf{\Delta}$ to denote the category with $\operatorname{obj}\mathbf{\Delta} = \{\,[n] = \{0, 1, \dots, n\} \mid n \in \mathbb{N}\,\}$ and $\hom_{\mathbf{\Delta}}([n],[m])$ the set of monotone maps. Then, for a category $\mathbf{D}$, the category of simplicial objects in $\mathbf{D}$, denoted $\bar{s}\mathbf{D}$, is defined to be $\mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}}}$.

We have special maps in the category $\mathbf{\Delta}$, namely $\delta_i\colon [n-1] \to [n]$, $\sigma_i\colon [n+1] \to [n]$ for $0 \le i \le n$ given by

$$\delta_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \ge i \end{cases} \qquad \sigma_i(j) = \begin{cases} j & \text{if } j \le i, \\ j-1 & \text{if } j > i \end{cases} \tag{2.1}$$

Then, if $K\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{D}$ is a functor, we define $K_n = K([n])$, $\partial_i = K(\delta_i)$ and $s_i = K(\sigma_i)$. The sequence of objects in $\mathbf{D}$, $K_0, K_1, K_2, \dots$ together with the $\partial_i\colon K_n \to K_{n-1}$, $s_i\colon K_n \to K_{n+1}$ for $0 \le i \le n$ satisfy

1. $\partial_i\partial_j = \partial_{j-1}\partial_i$ if $i < j$,

2. $s_i s_j = s_{j+1} s_i$ if $i \le j$,

3. $\partial_i s_j = s_{j-1}\partial_i$ if $i < j$,
   $\partial_j s_j = 1_{K_q} = \partial_{j+1} s_j$,
   $\partial_i s_j = s_j \partial_{i-1}$ if $i > j+1$.

Conversely, it can be proven that given objects K_n in $\mathbf{D}$ and maps ∂_i, s_i, satisfying the previous identities we obtain a simplicial object in $\mathbf{D}$.

The elements in K_q are called *q-simplices*. The maps ∂_i are called *face operators* and the s_i are called *degeneracy operators*. If $x \in K_q$ is such that $x = s_i y$ for some i

and some $y \in K_{q-1}$, we say that x is *degenerate*. We will denote by $\Delta[n]$ the simplicial set given by the contravariant functor $\hom_{\mathbf{\Delta}}(-, [n])$.

The category of bisimplicial objects in $\mathbf{D}$ is the category of simplicial objects in the category of simplicial objects in $\mathbf{D}$, that is $(\mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{\Delta}^{\mathrm{op}}}$. Using the exponential law for categories, we observe that this is the same as $\mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}$.

The diagonal functor $\operatorname{diag} \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ induces a functor $\mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}}}$ which we will also denote by diag, from the category of bisimplicial objects in $\mathbf{D}$ to the category of simplicial objects in $\mathbf{D}$. Note that any functor $F \colon \mathbf{C} \to \mathbf{D}$ gives a commutative diagram of categories and functors

$$\begin{array}{ccc}
\mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} & \xrightarrow{\operatorname{diag}} & \mathbf{C}^{\mathbf{\Delta}^{\mathrm{op}}} \\
{\scriptstyle F^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}}} \downarrow & & \downarrow {\scriptstyle F^{\mathbf{\Delta}^{\mathrm{op}}}} \\
\mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}} & \xrightarrow{\operatorname{diag}} & \mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}}}
\end{array}$$

Let k be a commutative ring with unit element. We will denote also by k the functor $\mathbf{Set} \to k\text{-}\mathbf{mod}$ that sends a set X to the free k-module generated by X. This functor extends to a functor from the category of simplicial sets to the category of simplicial k-modules, and also to a functor from the category of bisimplicial sets to the category of bisimplicial k-modules.

Lemma 2.1. *Let X be a bisimplicial set, and k be a commutative ring with unit element. Then $k(\operatorname{diag} X) \cong \operatorname{diag} kX$ in the category $\bar{s}k$-$\mathbf{mod}$.*

Proof. Apply the previous commutative square to the functor $k \colon \mathbf{Set} \to k\text{-}\mathbf{mod}$ □

Remark 2.2. The exponential law for functor categories, together with the commutativity of the product of two categories, allows us to deduce, for any category $\mathbf{D}$, that the category of G-simplicial objects in $\mathbf{D}$ (that is, $(\mathbf{D}^{\mathbf{\Delta}^{\mathrm{op}}})^{\mathbf{G}}$) is isomorphic to the category of simplicial G-objects in $\mathbf{D}$ (this is $(\mathbf{D}^{\mathbf{G}})^{\mathbf{\Delta}^{\mathrm{op}}}$). Also, clearly we have that if X is a bisimplicial G-set, then the simplicial set $\operatorname{diag}(X)$ has a canonical action of G.

We denote by Δ^n the standard n-simplex

$$\Delta^n = \{\, (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0 \,\} \tag{2.2}$$

Let e_i be the point in Δ^n with the i-th coordinate equal to 1, we call it the i-th vertex of Δ^n. For $f\colon [m] \to [n]$ we have a map $\Delta(f)\colon \Delta^m \to \Delta^n$ which is the linear map that sends the vertex e_i to the vertex $e_{f(i)}$.

Definition 2.3. The geometric realization $|X|$ of a simplicial set X is the topological space obtained as the quotient of $\tilde{X} = \coprod X_n \times \Delta^n$ (where X_n is given the discrete topology) by the equivalence relation generated by identifying points $(x, s) \in X_n \times \Delta^n$ and $(t, y) \in X_m \times \Delta^m$ if $y = X(f)x$ and $s = \Delta(f)(t)$ for some map $f\colon [m] \to [n]$. We denote the equivalence class of (x, s) by $[x, s]$.

In the case that X has an action of the group G, we notice that if we consider G acting trivially on all of the Δ^n, then G acts on $\tilde{X}$ and it preserves the equivalence relation, hence $|X|$ becomes a G-topological space by the action $g[x, s] = [gx, s]$. If K is a G-simplicial set, then $|K|$ is a G-space, and a map of G-simplicial sets $\phi\colon K_1 \to K_2$ induces an equivariant continuous map $|\phi|\colon |K_1| \to |K_2|$.

Remark 2.4. Note that the definition of geometric realization could also be used to define the geometric realization of a simplicial topological space.

Here we are mainly interested in the composition

$$\mathbf{SCat} \xrightarrow{N} \bar{s}\mathbf{Set} \xrightarrow{|-|} \mathbf{Top} \tag{2.3}$$

Theorem 2.5. *Let X be a bisimplicial G-set. Construct a simplicial G-topological space X^I by sending $[p]$ to the realization of the simplicial set $[q] \mapsto X_{pq}$. Similarly, construct another simplicial G-topological space X^{II} by sending $[q]$ to the realization of the simplicial set $[p] \mapsto X_{pq}$. Then we have a homeomorphism of G-topological spaces*

$$|\operatorname{diag}(X)| \cong_G |X^I| \cong_G |X^{II}| \tag{2.4}$$

Proof. We refer the reader to [GM96, p. 19]. The non-equivariant version of this theorem is proved there. The method used is to give a set Z and three equivalence relations on it so that by applying them in different orders we get $|\operatorname{diag}(X)|$, $|X^I|$ and $|X^{II}|$ and observing that the resulting spaces have to be homeomorphic. It is immediate to check that G preserves such equivalence relations, hence the isomorphisms given are equivariant. □

2.3 The Nerve Functor

Let **SCat** denote the category of small categories. Let $\mathbf{n}$ be the category with objects $\{1, \ldots, n\}$ and exactly one map $i \to j$ if $i \leq j$. Note that a monotone map $f\colon [n] \to [m]$ gives rise to a functor $f^*\colon \mathbf{n} \to \mathbf{m}$. Given a small category $\mathbf{C}$, we get a simplicial set $N(\mathbf{C})$, called the *nerve* of $\mathbf{C}$, as the functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ defined by $[n] \mapsto \hom_{\mathbf{SCat}}(\mathbf{n}, \mathbf{C})$ and a map $f\colon [n] \to [m]$ gives $\hom_{\mathbf{SCat}}(\mathbf{m}, \mathbf{C}) \to \hom_{\mathbf{SCat}}(\mathbf{n}, \mathbf{C})$ by precomposing with f^*. In other words, $N(\mathbf{C})$ is the simplicial set with q-simplices the diagrams in $\mathbf{C}$ of form

$$X_0 \to X_1 \to \cdots \to X_q \tag{2.5}$$

The i-th face of this simplex is defined by deleting X_i and composing maps as necessary and i-th degeneracy is obtained by replacing X_i by $1_{X_i}\colon X_i \to X_i$.

If $F\colon \mathbf{C}_1 \to \mathbf{C}_2$ is a functor between small categories, then

$$\begin{aligned}(X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \cdots X_{q-1} \xrightarrow{\phi_{q-1}} X_q) \mapsto \\ (FX_0 \xrightarrow{F\phi_0} FX_1 \xrightarrow{F\phi_1} FX_2 \cdots FX_{q-1} \xrightarrow{F\phi_{q-1}} FX_q)\end{aligned} \tag{2.6}$$

is a map $N(F)\colon N(C_1) \to N(C_2)$ of simplicial sets, since it commutes with faces and degeneracies.

In the case that $\mathbf{C}$ is a G-category, there is a natural structure of G-simplicial set in $N(\mathbf{C})$, that is also preserved by G-functors. If $\mathbf{C}$ is a small category, we denote $|N(\mathbf{C})|$ just by $|\mathbf{C}|$ and call it the geometric realization of the category $\mathbf{C}$. Similarly, if $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, we denote $|N(F)|$ by $|F|$.

Example 2.6. If G is a group and $\mathbf{C}$ is the category $\mathbf{G}$, then $|\mathbf{G}|$ is already known as the classifying space of G, denoted $\mathrm{B}G$.

In the next section we consider another example with a little more detail, which includes the case of a poset seen as a category.

2.4 Preordered Sets

A particular case that will be of interest of us is when the category $\mathbf{C}$ has the property that for any pair of objects A, B in $\mathbf{C}$ the hom set has only one element or is empty. This is exactly the same as a preordered set (see for example ([Mac71, page 11])), that is, a set with a binary relation $\leq$ that is both reflexive and transitive. If P is a preordered set, we can define a category $\mathbf{P}$ by putting $\operatorname{obj}\mathbf{P} = P$, and

$$\hom(x, y) = \begin{cases} x \leq y & \text{if } x \leq y \\ \emptyset & \text{if } x \nleq y \end{cases} \tag{2.7}$$

The composition is given by $(x \leq y, y \leq z) \mapsto x \leq z$. A G-action on the category $\mathbf{P}$ corresponds to an action of G on P in such a way that $x \leq y$ in P and $g \in G$ imply $gx \leq gy$. If Q is another preordered set, then a functor $F\colon \mathbf{P} \to \mathbf{Q}$ corresponds to an equivariant order-preserving map $F\colon P \to Q$. The nerve of $\mathbf{P}$ is the simplicial set whose q-simplices can be identified with chains $x_0 \leq x_1 \leq \cdots \leq x_q$ of length q in P. The non-degenerate q-cells are the chains where none of the inequalities is an equality. If the relation $\leq$ is in addition antisymmetric, that is, if $(P, \leq)$ is actually a poset, then the chains

$$x_0 < x_1 < \cdots < x_q \tag{2.8}$$

give the q-simplices of a simplicial complex, and so $|\mathbf{P}|$ can be identified with the geometric realization of it. This is the simplicial complex we referred to in the Introduction. It is usually called the order complex of P and denoted $\Delta(P)$.

Chapter 3

Equivariant Homotopy Type of Categories and Simplicial Sets

We present here some technical results which will be used later in our analysis of the structure of various examples. These results are well-known in their non-equivariant form, and some also in case the categories are posets [TW91]. The equivariant form of Quillen's Theorem A (Theorem 3.10 here) is new in the generality presented here, as also is the equivariant form of Thomason's theorem 3.23.

3.1 Basic Theorems

Definition 3.1. If X and Y are G-spaces, a *G-homotopy* from X to Y is a continuous map $H\colon X \times [0,1] \to Y$ such that $H(gx,t) = gH(x,t)$ for all $g \in G$, $x \in X$ and $t \in [0,1]$. If K and L are G-simplicial sets and $\phi, \psi\colon K \to L$ are G-maps, we say that ϕ is weakly G-homotopic to ψ if there is a G-homotopy from $|K|$ to $|L|$ such that $H(x,0) = |\phi|(x)$ and $H(x,1) = |\psi|(x)$.

In the same way, we say that two G-functors between G-categories $F_1, F_2\colon \mathbf{C} \to \mathbf{D}$ are weakly G-homotopic if there is a G-homotopy from $|\mathbf{C}|$ to $|\mathbf{D}|$ such that $H(x,0) = |F_1|(x)$ and $H(x,1) = |F_2|(x)$. Also we say that two G-categories $\mathbf{C}$ and $\mathbf{D}$ are weakly G-homotopy equivalent if $|\mathbf{C}|$ and $|\mathbf{D}|$ are G-homotopy equivalent.

Remember that $\Delta[1]$ is the simplicial set $\hom_{\mathbf{\Delta}}(-,[1])$. Let us denote by 0 its subsimplicial set of maps that have images contained in $\{0\}$ and by 1 the subsimplicial set of maps that have images contained in $\{1\}$. Hence 0 and 1 are both simplicial sets with only one simplex on each dimension.

Definition 3.2. If K and L are G-simplicial sets and $\phi, \psi \colon K \to L$ are G-maps, we say that ϕ is strongly (or simplicially) G-homotopic to ψ if there is a G-simplicial set map $H \colon K \times \Delta[1] \to L$ such that H restricted to $K \times 0$ can be identified with ϕ and H restricted to $K \times 1$ can be identified with ψ.

Similarly, we say that two G-functors between G-categories $F_1, F_2 \colon \mathbf{C} \to \mathbf{D}$ are strongly G-homotopic if $N(F_1)$ and $N(F_2)$ are strongly G-homotopic.

Theorem 3.3. *If K and L are G-simplicial sets, we have a G-homeomorphism*

$$|K \times L| \cong_G |K| \times |L| \tag{3.1}$$

if the topology on the right side is taken to be compactly generated.

See for example [GJ97, Proposition 2.4].

Corollary 3.4. *If K and L are G-simplicial sets and $\phi, \psi \colon K \to L$ are strongly G-homotopic G-maps, they are also weakly G-homotopic maps.*

For the rest of this section, $\mathbf{C}$ and $\mathbf{D}$ will denote G-categories.

We say that $\mathbf{C}$ is G-contractible if $\mathbf{C}$ is weakly G-homotopy equivalent to a point.

Lemma 3.5. *(Compare with* [BK72, page 292]*) A natural transformation $\eta \colon F \to F'$ between the G-functors $F, F' \colon \mathbf{C} \to \mathbf{D}$ induces a strong G-homotopy between F and F'.*

Proof. The natural transformation allows us to define a G-functor $\tau \colon \mathbf{C} \times \{0 < 1\} \to \mathbf{D}$ by $(X, 0) \mapsto F(X)$, $(X, 1) \mapsto F'(X)$. Since $N(\mathbf{C} \times \{0 < 1\}) \cong_G N(\mathbf{C}) \times \Delta[1]$, we have that $N(\tau)$ is a G-strong homotopy, $\square$

Note that in the previous lemma we do not ask any kind of equivariant behavior to the natural transformation η.

Corollary 3.6. *If the G-functor $F\colon \mathbf{C} \to \mathbf{D}$ is left adjoint to the G-functor $F'\colon \mathbf{D} \to \mathbf{C}$, then F and F' are strong G-homotopy equivalences.*

Proof. We apply Lemma 3.5 using the unit and counit of the adjunction $1_{\mathbf{C}} \to F'F$ and $FF' \to 1_{\mathbf{D}}$. $\square$

Corollary 3.7. *If $\mathbf{C}$ is a G-category with an initial object X fixed by G, then $\mathbf{C}$ is strongly G-contractible.*

Proof. Let $\mathbf{X}$ be the category with only one object X and only one morphism. Define a functor $S\colon \mathbf{C} \to \mathbf{X}$ by sending every object to X, and all maps to the identity. Let $T\colon \mathbf{X} \to \mathbf{C}$ include the object X in $\mathbf{C}$. Given that X is a fixed object, both S and T are G-functors. Then ST is the identity in $\mathbf{X}$, and we prove that TS is homotopic to the identity because for any object C in $\mathbf{C}$ we can take the unique map $C \to X$ as defining the component of a natural transformation η_C between TS and the identity. $\square$

The following criterion for a homotopy equivalence to be an equivariant homotopy equivalence is often useful.

Theorem 3.8. *Let X and Y be G-CW-complexes and $\phi\colon X \to Y$ a G-equivariant cellular map. Then ϕ is a G-homotopy equivalence if and only if $\phi^H\colon X^H \to Y^H$ is a homotopy equivalence for each subgroup H of G.*

Proof. See [Bre67], Sect. II. $\square$

Definition 3.9. Following Quillen ([Qui72, page 93]), we make the following definition: if $F\colon \mathbf{C} \to \mathbf{D}$ is a functor and $D \in \operatorname{obj}\mathbf{D}$, let F/D denote the category with objects (C, v) with $C \in \operatorname{obj}\mathbf{C}$ and $v\colon FC \to D$. A morphism from (C, v) to (C', v') is a map $w\colon C \to C'$ such that $v'F(w) = v$. We similarly define the category $D\backslash F$ with objects the pairs (C, u) with $u\colon D \to FC$.

We remark that if F is a G-functor, then both F/D and $D\backslash F$ have an action of the stabilizer G_D induced by the action of G on $\mathbf{C}$ and $\mathbf{D}$, namely $g(C, v) = (gC, gv)$. We now prove a new result, the equivariant form of Quillen's Theorem, which is probably the most powerful technique available to prove that a functor induces a weak equivariant homotopy equivalence of categories. It is at the same time a generalization of Quillen's Theorem A from [Qui72] and of (1.4) from [TW91].

Theorem 3.10. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a G-functor. If the category $D\backslash F$ is weakly G_D-contractible for every object D of $\mathbf{D}$, then F is a weak G-homotopy equivalence.*

Proof. We follow the argument of (1.4) from [TW91]. We apply Lemma 3.8 to $|F|\colon |\mathbf{C}| \to |\mathbf{D}|$. Let $H \leq G$ and consider $|F|^H\colon |\mathbf{C}|^H \to |\mathbf{D}|^H$. We have that $|\mathbf{C}|^H = |\mathbf{C}^H|$, where $\mathbf{C}^H$ denotes the full subcategory of $\mathbf{C}$ with objects the ones fixed by H. Hence it is enough to prove that $F^H\colon \mathbf{C}^H \to \mathbf{D}^H$ is an ordinary homotopy equivalence, where F^H denotes the restriction of F. We therefore apply the non-equivariant form of Quillen's theorem to F^H. Let D be an object in $\mathbf{D}^H$ so that $H \leq G_D$, and consider $D\backslash(F^H)$. It is clear that this can be identified with $(D\backslash F)^H$, and the G_D-contraction of $D\backslash F$ restricts to give a contraction of the fixed points, $\square$

3.2 The Homotopy Colimit

The homotopy colimit of a functor $F\colon \mathbf{C} \to \bar{s}\mathbf{Set}$ was studied in [BK72] and has been proved very useful in some contexts in algebra ([Dwy97, Dwy98a, Dwy98b]) and combinatorics ([WZŽ98]). In the case that F actually takes values in the category of G-simplicial sets, then $\operatorname{hocolim} F$ will be an object in the same category and so it will have defined a G-action. However, we will see that for an action of G to be defined on $\operatorname{hocolim} F$ in the more general situation it is also sufficient to let $\mathbf{C}$ be a G-category and F be compatible with the action on $\mathbf{C}$. The following definition makes this precise.

Definition 3.11. Let $\mathbf{C}$ be a G-category and $F\colon \mathbf{C} \to \mathbf{D}$ a functor. ($\mathbf{D}$ is now not necessarily a G-category). Suppose that for each $g \in G$, $X \in \operatorname{obj} \mathbf{C}$ there is a map $\eta_{g,X}\colon F(X) \to F(gX)$ for $X \in \operatorname{obj} \mathbf{C}$, $g \in G$ such that

1. $\eta_{1,X} = 1_{F(X)}$ for all $X \in \operatorname{obj} \mathbf{C}$

2. the following diagram is commutative for $X \in \operatorname{obj} \mathbf{C}$, $g_1, g_2 \in G$:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_{g_2,X}"] \arrow[dr, "\eta_{g_1 g_2,X}"'] & F(g_2 X) \arrow[d, "\eta_{g_1,g_2 X}"] \\ & F(g_1 g_2 X) \end{tikzcd} \tag{3.2}$$

3. the following diagram is commutative for $g \in G$ and $f\colon X \to Y$ a map in $\mathbf{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_{g,X}"'] & F(Y) \arrow[d, "\eta_{g,Y}"] \\ F(gX) \arrow[r, "F(gf)"] & F(gY) \end{tikzcd} \tag{3.3}$$

(this is, for a fixed $g \in G$, $\{\eta_{g,X}\}_{X \in \operatorname{obj} \mathbf{C}}$ is a natural transformation between F and Fg).

Then, we call η an *action by natural transformations* on the functor F. Note that by 1. and 2., the maps $\eta_{g,X}$ have to be isomorphisms.

Remark 3.12. Observe that if $F\colon \mathbf{C} \to \mathbf{D}$ is a functor with action by natural transformations η and T is any functor $T\colon \mathbf{D} \to \mathbf{E}$, then $T\eta$ is an action by natural transformations on $TF\colon \mathbf{C} \to \mathbf{E}$. Note also that each object FX obtains an action of the subgroup G_X by $\eta_{g,X}\colon FX \to FX$ in such a way that if $\phi\colon X \to Y$ is a map in $\mathbf{C}$, then $F\phi\colon FX \to FY$ is $G_X \cap G_Y$-equivariant.

Definition 3.13. ([BK72, page 337]) Let $\mathbf{C}$ be a category and $F\colon \mathbf{C} \to \mathbf{D}$ a functor. We define the *simplicial replacement* of F, which is the simplicial object $K(F, \mathbf{C})$ in $\mathbf{D}$ such that

$$K(F, \mathbf{C})_n = \coprod_{\overline{X}} FX_0 \tag{3.4}$$

where the coproduct is taken over all the n-simplices $\overline{X} = X_0 \xrightarrow{\phi_0} X_1 \to \cdots \to X_{n-1} \to X_n$ of $N(\mathbf{C})$. Faces and degeneracies are induced by the maps

$$d_0\colon FX_0 \xrightarrow{F\phi_0} FX_1 \tag{3.5}$$

$$d_j\colon FX_0 \xrightarrow{1} FX_0, j > 0 \tag{3.6}$$

$$s_j\colon FX_0 \xrightarrow{1} FX_0 \tag{3.7}$$

where in (3.5) the domain is indexed by $\overline{X}$ and the range by $d_0^{N(\mathbf{C})}\overline{X}$, in (3.6) the domain is indexed by $\overline{X}$ and the range by $d_j^{N(\mathbf{C})}\overline{X}$ and in (3.7), the domain is indexed by $\overline{X}$ and the range by $s_j^{N(\mathbf{C})}\overline{X}$

Our definition of simplicial replacement differs slightly from that in [BK72]. We follow [GZ67, page 153] and [Dwy98a, page 20]. As an example of the usefulness of this concept, we have that if $\mathbf{D} = \mathbf{Ab}$, the homology of the simplicial abelian group $K(F, \mathbf{C})$ gives the derived functors of the functor $\operatorname{colim}\colon \mathbf{Ab}^{\mathbf{C}} \to \mathbf{Ab}$ at F ([Qui72, page 91]). The correspondance $F \mapsto K(F, \mathbf{C})$ is actually functorial on F: if $\varepsilon\colon F_1 \to F_2$ is a natural transformation, then there is a simplicial map $K(F_1, \mathbf{C}) \to K(F_2, \mathbf{C})$ induced by $\varepsilon_{X_0}\colon F_1X_0 \to F_2X_0$ on the term indexed by $\overline{X}$. Commutativity with faces and degeneracies is clear, the only nontrivial part is commutativity with d_0, which is a consequence of the commutativity of

$$\begin{array}{ccc} F_1X_0 & \xrightarrow{\varepsilon_{X_0}} & F_2X_0 \\ {\scriptstyle F_1\phi_0}\downarrow & & \downarrow{\scriptstyle F_2\phi_0} \\ F_1X_1 & \xrightarrow{\varepsilon_{X_1}} & F_2X_1 \end{array}$$

for any map $\phi_0\colon X_0 \to X_1$. Hence $K(-, \mathbf{C})$ is a functor $\mathbf{D}^{\mathbf{C}} \to \bar{s}\mathbf{D}$.

We now use Definition 3.13 to define homotopy colimits.

Definition 3.14. ([Dwy98a, Definition 4.13.]) Let $F\colon \mathbf{C} \to \bar{s}\mathbf{D}$ be a functor, so that $K(F, \mathbf{C})$ is a bisimplicial object in $\mathbf{D}$. We define the *homotopy colimit* of F, denoted $\operatorname{hocolim} F$, by

$$\operatorname{hocolim} F = \operatorname{diag} K(F, \mathbf{C}) \tag{3.8}$$

In particular, if $\mathbf{D} = \mathbf{Set}$, then $\operatorname{hocolim} F$ is a simplicial set with p-simplices:

$$(\operatorname{hocolim} F)_p = \{\, (\overline{X}, y) \mid \overline{X} = (X_0 \xrightarrow{\phi_0} X_1 \to \cdots \to X_{p-1} \to X_p) \in N(\mathbf{C})_p, y \in F(X_0)_p \,\} \tag{3.9}$$

and faces and degeneracies given by:

$$d_0(\overline{X}, y) = \left(d_0^{N(\mathbf{C})}\overline{X}, d_0^{F(X_1)}F(\phi_0)(y) = F(\phi_0)(d_0^{F(X_0)}y)\right) \tag{3.10}$$

$$d_i(\overline{X}, y) = (d_i^{N(\mathbf{C})}\overline{X}, d_i^{F(X_0)}y), \quad i > 0 \tag{3.11}$$

$$s_i(\overline{X}, y) = (s_i^{N(\mathbf{C})}\overline{X}, s_i^{F(X_0)}y) \tag{3.12}$$

Proposition 3.15. *Let $F\colon \mathbf{C} \to \bar{s}\mathbf{Set}$ be a functor. Then*

$$k(\operatorname{hocolim} F) \cong \operatorname{hocolim}(kF). \tag{3.13}$$

Proof. The functor k preserves coproducts and diagonals, by Lemma 2.1. □

If A is a bisimplicial k-module, there is a spectral sequence

$$E^2_{pq} = H_p(H_q A) \Rightarrow H_{p+q}(\operatorname{diag} A) \tag{3.14}$$

(see [GJ97, Chapter IV]). If the bisimplicial k-module is $K(L, \mathbf{C})$ for a functor $L\colon \mathbf{C} \to \bar{s}k\text{-}\mathbf{mod}$, this corresponds to (compare [GZ67, page 153] and [Dwy98a, page 21]) the spectral sequence of Bousfield and Kan

$$E^2_{pq} = \operatorname{colim}_p(H_q L) \Rightarrow H_{p+q}(\operatorname{hocolim} L) \tag{3.15}$$

where $H_q L$ is a functor $k\text{-}\mathbf{mod}^{\mathbf{C}} \to k\text{-}\mathbf{mod}$ sending X to $H_q(LX)$ and colim_p is the p-th left derived functor of $\operatorname{colim}\colon k\text{-}\mathbf{mod}^{\mathbf{C}} \to k\text{-}\mathbf{mod}$.

We now let the G-action come into play:

Proposition 3.16. *Let $\mathbf{C}$ be a G-category and η an action by natural transformations on $F\colon \mathbf{C} \to \mathbf{D}$. Then $K(F, \mathbf{C})$ can be given a natural structure of G-simplicial object.*

Proof. Note that as an object in $\mathbf{D}$, $K(F, \mathbf{C})_n$ can be identified with $K(Fg, \mathbf{C})_n$ because g only permutes the chains $\overline{X} \in N(\mathbf{C})_n$ and the set of chains of the form $\overline{X}$ is the same as the set of chains of the form $g\overline{X}$, possibly taken in a different order). Hence for each g, the natural transformation $\eta_g\colon F \to Fg$ induces a simplicial map $K(F, \mathbf{C}) \to K(Fg, \mathbf{C}) \equiv K(F, \mathbf{C})$ acting as $\eta_{g,X_0}\colon FX_0 \to F(gX_0)$ on the term X_0 corresponding to $\overline{X}$. This evidently gives an action of G. □

Corollary 3.17. *If $\mathbf{C}$ is a G-category, $F\colon \mathbf{C} \to \bar{s}\mathbf{D}$ a functor and η an equivariant automorphism of F, then* hocolim F *is a G-simplicial object in $\mathbf{D}$, with action in the p-simplices given by:*

$$g(X_0 \xrightarrow{\phi_0} X_1 \to \cdots \to X_p, y) = (gX_0 \xrightarrow{g\phi_0} gX_1 \cdots \to gX_p, \eta_{g,X_0}(y)) \tag{3.16}$$

for $y \in (FX_0)_p$.

Proof. This is precisely the G-action on $\operatorname{diag} K(F, \mathbf{C})$ induced from the G-action on $K(F, \mathbf{C})$. $\square$

Lemma 3.18. *For a G-simplicial set X, we have that $|X|^G$ is naturally homeomorphic to $|X^G|$.*

Proof. This follows from [GJ97, Chapter I, Proposition 2.4.], which says that the functor $|-|$ preserves finite limits, and that X^G is the limit of the functor $\mathbf{G} \to \bar{s}\mathbf{Set}$ that gives X its action of G. $\square$

Theorem 3.19. *Let $\phi\colon X \to Y$ a map of bisimplicial G-sets. Suppose that for all p we have that $\phi_p\colon X_p \to Y_p$ is a weak G-homotopy equivalence. Then* $\operatorname{diag}\phi$ *is a weak G-homotopy equivalence.*

Proof. The non-equivariant version of this theorem is [GJ97, Chapter IV, Proposition 1.9.]. We will use Lemma 3.8. To prove that $|\operatorname{diag}(\phi)|$ is a G-homotopy equivalence we need to check that $|\operatorname{diag}(\phi)|^H$ is an ordinary homotopy equivalence. By the previous lemma, we just need to verify that $\operatorname{diag}(\phi^H)\colon \operatorname{diag}(X^H) \to \operatorname{diag}(Y^H)$ is a weak homotopy equivalence since clearly $\operatorname{diag}(Y)^H$ can be identified with $\operatorname{diag}(Y^H)$. By the non-equivariant version of Theorem 3.19, it is enough to prove that $|(\phi^H)_p|\colon |(X^H)_p| \to |(Y^H)_p|$ is a homotopy equivalence. But this follows from the fact that $|(X^H)_p| = |X_p|^H$ (similarly for Y) and that the equivariant homotopy equivalence $|\phi_p|\colon |X_p| \to |Y_p|$ restricts to fixed points. $\square$

We will prove now the equivariant homotopy invariance of the homotopy colimit. In this theorem, we prove that if we have two functors that have equivariant homotopy equivalent values by a natural transformation, then their homotopy colimits are equivariant homotopy equivalent. The analogous statement is not true if we replace homotopy colimits by the ordinary colimit. The nonequivariant form of this theorem can be found in [BK72, page 335].

Theorem 3.20 (Homotopy Invariance of the Homotopy Colimit). *Let*

$$F, F'\colon \mathbf{C} \to \bar{s}\mathbf{Set} \tag{3.17}$$

be two functors with action by natural transformations η, η' respectively. Let $\varepsilon\colon F \to F'$ be a natural transformation such that each $\varepsilon_X\colon FX \to F'X$ is a weak G_X-homotopy equivalence Suppose that the following diagram commutes for all $g \in G$, $X \in \operatorname{obj}\mathbf{C}$

$$\begin{tikzcd} FX \arrow[r, "\varepsilon_X"] \arrow[d, "\eta_{g,X}"'] & F'X \arrow[d, "\eta'_{g,X}"] \\ F(gX) \arrow[r, "\varepsilon_{gX}"] & F'(gX) \end{tikzcd}$$

Then we may construct an equivariant map

$$\operatorname{hocolim} F \to \operatorname{hocolim} F' \tag{3.18}$$

which is an equivariant weak homotopy equivalence.

Proof. The natural transformation $\varepsilon\colon F \to F'$ induces a map $K(F, \mathbf{C}) \to K(F', \mathbf{C})$. This map is given on (p, q)-simplices $\bigsqcup_{\overline{X}}(FX_0)_q \to \sqcup_{\overline{X}}(F'X_0)_q$ by sending $(\overline{X}, y)$ to $(\overline{X}, \varepsilon_{X_0}(y))$ for $\overline{X} \in N(\mathbf{C})_p$. We check it is equivariant. For a chain $\overline{X} = X_0 \to \cdots \to X_p$ we denote $gX_0 \to \cdots \to gX_p$ by $\overline{gX}$.

$$\begin{aligned} \varepsilon_{X_0}\left(g(\overline{X}, y)\right) &= \varepsilon_{X_0}\left(\overline{gX}, \eta_{g,X_0}(y)\right) \\ &= \left(\overline{gX}, \varepsilon_{gX_0}\eta_{g,X_0}(y)\right) \\ &= \left(\overline{gX}, \eta'_{g,X_0}\varepsilon_{X_0}(y)\right) \\ &= g\left(\overline{X}, \varepsilon_{X_0}(y)\right) \\ &= g\varepsilon_{X_0}(\overline{X}, y) \end{aligned}$$

Hence a simplicial G-map $\operatorname{hocolim} F \to \operatorname{hocolim} F'$ is obtained after taking diagonals. To check it is a weak G-homotopy equivalence we apply Theorem 3.19 to the map $K(F, \mathbf{C}) \to K(F', \mathbf{C})$. Let E_p be a set of representatives for the orbits of the action of G on $N(\mathbf{C})_p$. The map $|K(F, \mathbf{C})_p| \to |K(F', \mathbf{C})_p|$ can then be written as

$$\bigsqcup_{\overline{Y} \in E_p} \operatorname{ind}_{G_{\overline{Y}}}^G |FY_0| \to \bigsqcup_{\overline{Y} \in E_p} \operatorname{ind}_{G_{\overline{Y}}}^G |F'Y_0| \tag{3.19}$$

(see [tD87, page 32] for the definition and properties of induced topological spaces). Since by hypothesis, each $|\varepsilon_{Y_0}|\colon |FY_0| \to |F'Y_0|$ is a G_{Y_0}-homotopy equivalence, given that $G_{\overline{Y}} \subseteq G_{Y_0}$ it is also a $G_{\overline{Y}}$-homotopy equivalence. Therefore the map (3.19) is a coproduct of G-homotopy equivalences, hence a G-homotopy equivalence, as we wanted to prove. $\square$

Remark 3.21. In the proof of the previous theorem, we needed the following facts, which are easy to prove. Let H be a subgroup of G. Then

1. If $f\colon X \to Y$ is a G-homotopy equivalence, then the restriction of f to H is an H-homotopy equivalence.

2. If $f\colon X \to Y$ is an H-homotopy equivalence, then $\operatorname{ind}_H^G f\colon \operatorname{ind}_H^G X \to \operatorname{ind}_H^G Y$ is a G-homotopy equivalence.

Let $F\colon \mathbf{C} \to \mathbf{SCat}$ a functor with an action by natural transformations η from a G-category $\mathbf{C}$. By Remark 3.12, we have that the composition of F with the nerve functor $N\colon \mathbf{SCat} \to \bar{s}\mathbf{Set}$ has $N(\eta)$ as an action by natural transformations. Hence $\operatorname{hocolim} N(F)$ is a G-simplicial set. Our aim now is to obtain a G-category defined in terms of F which is weakly G-homotopy equivalent to $\operatorname{hocolim} N(F)$, therefore simplifying the task of identifying the homotopy type of $|\operatorname{hocolim} N(F)|$.

Definition 3.22. Let $F\colon \mathbf{C} \to \mathbf{SCat}$ be a functor. We define a category $\operatorname{Gr}(F)$ with objects the pairs (X, a) with $X \in \operatorname{obj}\mathbf{C}$, $a \in \operatorname{obj} F(X)$. A map $(X, a) \to (Y, b)$ is given by a pair (f, u) such that $f\colon X \to Y$ is a map in $\mathbf{C}$ and $u\colon F(f)(a) \to b$ is a map in the category $F(Y)$. The category $\operatorname{Gr}(F)$ is called the *Grothendieck Construction* on F.

Now, if $F\colon \mathbf{C} \to \mathbf{SCat}$ is a functor from a G-category $\mathbf{C}$ with an action by natural transformations η, we have that $\operatorname{Gr}(F)$ is a G-category with action on objects given by

$$g(X, a) = (gX, \eta_{g,X}(a)) \tag{3.20}$$

and on maps by

$$g\left((X, a) \xrightarrow{(f,u)} (Y, b)\right) = (gf, \eta_{g,Y}(u)) \tag{3.21}$$

The nonequivariant form of the next theorem is Theorem 1.2 of [Tho79].

Theorem 3.23. *Let* $\mathbf{C}$ *be a* G*-category and* $F\colon \mathbf{C} \to \mathbf{SCat}$ *a functor with an equivariant automorphism* η*. If we define on* $\operatorname{hocolim} N(F)$ *a* G*-action by* $N(\eta)$ *and on* $\mathrm{Gr}(F)$ *by* η*, then there is a weak* G*-homotopy equivalence*

$$\operatorname{hocolim} N(F) \to N\left(\mathrm{Gr}(F)\right) \tag{3.22}$$

Proof. We need to check that the maps defined by Thomason are equivariant. Notice that our notation differs slightly from Thomason's. We define $\Phi_i = \phi_i \cdots \phi_0$. We have the map of simplicial sets

$$\lambda\colon\ \operatorname{hocolim} N(F) \to N(\mathrm{Gr}(F)) \tag{3.23}$$

by sending

$$(X_0 \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{p-1}} X_p, a_0 \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{p-1}} a_p) \tag{3.24}$$

to

$$\begin{aligned}(X_0, a_0) &\xrightarrow{(\phi_0, F(\phi_0)(\alpha_0))} (X_1, F(\phi_0)(a_1)) \xrightarrow{(\phi_1, F(\Phi_1)(\alpha_1))} (X_2, F(\Phi_1)(a_2)) \\ &\to \cdots \to (X_{p-1}, F(\Phi_{p-2})(a_{p-1})) \xrightarrow{(\phi_{p-1}, F(\Phi_{p-1})(\alpha_{p-1}))} (X_p, F(\Phi_{p-1})(a_p))\end{aligned} \tag{3.25}$$

Thomason proved that λ is a homotopy equivalence, we want to prove that both λ and its homotopic inverse are equivariant. We first prove that λ is equivariant. We have

$$\begin{aligned}&\lambda(g(X_0 \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{p-1}} X_p, a_0 \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{p-1}} a_p)) \\ &= \lambda(gX_0 \xrightarrow{g\phi_0} \cdots \xrightarrow{g\phi_{p-1}} gX_p, \eta_{g,X_0}(a_0) \xrightarrow{\eta_{g,X_0}(\alpha_0)} \cdots \xrightarrow{\eta_{g,X_0}(\alpha_{p-1})} \eta_{g,X_0}(a_p)) \\ &= (gX_0, \eta_{g,X_0}(a_0)) \xrightarrow{(g\phi_0, F(g\phi_0)(\eta_{g,X_0}(\alpha_0)))} (gX_1, F(g\phi_0)\eta_{g,X_0}(a_1)) \\ &\quad \cdots (gX_{p-1}, F(g\Phi_{p-2})\eta_{g,X_0}(a_{p-1})) \xrightarrow{(g\phi_{p-1}, F(g\Phi_{p-1})(\eta_{g,X_0}(\alpha_{p-1})))} (gX_p, F(g\Phi_{p-1})\eta_{g,X_0}(a_p)) \\ &= (gX_0, \eta_{g,X_0}(a_0)) \xrightarrow{(g\phi_0, \eta_{g,X_1}(F(\phi_0)\alpha_0))} (gX_1, \eta_{g,X_1}(F(\phi_0)a_1)) \\ &\quad \cdots (gX_{p-1}, \eta_{g,X_{p-1}}(F(\Phi_{p-2})(a_{p-1}))) \xrightarrow{(g\phi_{p-1}, \eta_{g,X_{p-1}}F(\Phi_{p-1})(\alpha_{p-1}))} (gX_p, \eta_{g,X_{p-1}}F(\Phi_{p-1})(a_p)) \\ &= g\lambda(X_0 \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{p-1}} X_p, a_0 \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_{p-1}} a_p)\end{aligned} \tag{3.26}$$

This proves λ equivariant. Now, associated to the functor F we define a functor $\tilde{F}\colon \mathbf{C} \to \mathbf{SCat}$, such that for $X \in \operatorname{obj} \mathbf{C}$, $\tilde{F}(X)$ is the category with objects the pairs (l, a), with $l\colon Y \to X$ a map in $\mathbf{C}$ and $a \in \operatorname{obj} F(Y)$. A map $(k, z)\colon (l, a) \to (l', a')$ is composed by a map $k\colon Y \to Y'$ in $\mathbf{C}$ such that $l = l'k$ and $z\colon F(k)(a) \to a'$ is a map in the category $F(Y')$. The composition in $\tilde{F}(X)$ is given by $(k_1, z_1)(k_2, z_2) = (k_1k_2, z_1F(k_2)(z_2))$. A map $f\colon X \to Y$ in $\mathbf{C}$ gives a functor $\tilde{F}(f)$ defined on objects as $\tilde{F}(f)(l, a) = (fl, a)$ and on maps as $\tilde{F}(f)(k, z) = (k, z)$. From the action by natural transformations η on F, we define an action by natural transformations $\tilde{\eta}$ on $\tilde{F}$ by $\tilde{\eta}_{g,X}(l, a) = (gl, \eta_{g,Y}(a))$ and $\tilde{\eta}_{g,X}(k, z) = (gk, \eta_{g,Y'}(z))$.

We now prove there is a weak G-homotopy equivalence

$$\lambda_1\colon \operatorname{hocolim} N\tilde{F} \to \operatorname{hocolim} NF. \tag{3.27}$$

For each $X \in \operatorname{obj} \mathbf{C}$ there is a functor $\tilde{F}(X) \to F(X)$ defined on objects as $(l, a) \to F(l)(a)$ and on maps by $(k, z) \mapsto F(l')(z)$. This functor has a right adjoint $F(X) \to \tilde{F}(X)$ defined by $a \mapsto (1_X, a)$. We can verify that these are actually G_X-functors. By Corollary 3.6, we obtain that $N\tilde{F}(X) \to NF(X)$ is a strong G_X-homotopy equivalence. Also we have in this way a natural transformation $\tilde{F} \to F$, hence we have a natural transformation $N\tilde{F} \to NF$. It is straightforward to prove the commutativity required in Theorem 3.19 (to check it for the nerve of categories is enough to check it for objects and maps), from where the claim of the existence of the G-homotopy equivalence (3.27) follows.

We now will produce a weak equivariant homotopy equivalence

$$\lambda_2\colon \operatorname{hocolim} N\tilde{F} \to N(\operatorname{Gr}(F)) \tag{3.28}$$

As Thomason indicates, a p-simplex in $N\tilde{F}(X)$

$$(l_0, a_0) \xrightarrow{(k_0,z_0)} (l_1, a_1) \to \cdots \to (l_{p-1}, a_{p-1}) \xrightarrow{(k_{p-1},z_{p-1})} (l_p, a_p) \tag{3.29}$$

corresponds to a p-simplex in $\operatorname{Gr}(F)$

$$(Y_0, a_0) \xrightarrow{(k_0,z_0)} (Y_1, a_1) \to \cdots \to (Y_{p-1}, a_{p-1}) \xrightarrow{(k_{p-1},z_{p-1})} (Y_p, a_p) \tag{3.30}$$

together with the map $l_p\colon Y_p \to X$.

Hence $K(\tilde{F}, \mathbf{C})$ has as (p, q) simplices the expressions of the form

$$X_0 \to \cdots \to X_p, \quad Y_q \to X_0, \quad (Y_0, a_0) \to \cdots \to (Y_q, a_q) \tag{3.31}$$

and the map λ_2 will send a (q, q) simplex of this form to the q-simplex $(Y_0, a_0) \to \cdots \to (Y_q, a_q)$ in $N(\mathrm{Gr}(F))$. This is clearly equivariant.

Consider $N(\mathrm{Gr}(F))$ as a bisimplicial set constant in the p-direction, that is

$$N(\mathrm{Gr}(F))_{pq} = N(\mathrm{Gr}(F))_q. \tag{3.32}$$

Then the map λ_2 can be identified with $\mathrm{diag}(\Lambda)$ for an obvious bisimplicial G-map $\Lambda\colon K(\tilde{F}, \mathbf{C}) \to N(\mathrm{Gr}(F))_{**}$. From Theorem 2.5 we know that $|\operatorname{diag} N(\mathrm{Gr}(F))_{**}| \cong_G |N(\mathrm{Gr}(F))|$. So, by Theorem 3.19, we just need to prove that $\Lambda_q K(\tilde{F}, \mathbf{C})_{*q} \to N(\mathrm{Gr}(F))_{*q}$ is a weak homotopy equivalence. From (3.31), we observe that such map can be expressed as a coproduct of simplicial maps $N(Y_q \backslash \mathbf{C}) \to \Delta[0]$, taken over the points of $N(\mathrm{Gr}(F))_q$. Since geometric realization commutes with coproducts, we are to prove that the map

$$\coprod_{(Y_0,a_0)\to\cdots\to(Y_q,a_q)} |N(Y_q \backslash \mathbf{C})| \to \coprod_{(Y_0,a_0)\to\cdots\to(Y_q,a_q)} |\Delta[0]| \tag{3.33}$$

is a G-homotopy equivalence. Let E_q be a set of representatives for the action of G on $N(\mathrm{Gr}(F))_q$. Then we can write the spaces and the map in (3.33) as

$$\coprod_{i \in E_q} \mathrm{ind}_{G_i}^G |N(Y_q \backslash \mathbf{C})| \to \coprod_{i \in E_q} \mathrm{ind}_{G_i}^G |\Delta[0]| \tag{3.34}$$

The map $|N(Y_q \backslash \mathbf{C})| \to |\Delta[0]|$ is a G_i-homotopy equivalence by Corollary 3.7, because 1_{Y_q} is an initial object in $Y_q \backslash \mathbf{C}$ fixed by G_i. Hence (3.34) is a coproduct of G-homotopy equivalences, from where our claim that (3.28) is a G-homotopy equivalence follows.

We finally prove that $\lambda \cdot \lambda_1$ is strongly G-homotopic to λ_2. Note that $\lambda \cdot \lambda_1$ sends the p-simplex of $\operatorname{hocolim} N\tilde{F}$

$$\begin{aligned}(X_0 \xrightarrow{\phi_0} X_1 \to \cdots X_{p-1} \xrightarrow{\phi_{p-1}} X_p,& \\ (l_0, a_0) \xrightarrow{(k_0, z_0)} (l_1, a_1) \to \cdots \to (l_{p-1}, a_{p-1}) &\xrightarrow{(k_{p-1}, z_{p-1})} (l_p, a_p))\end{aligned} \tag{3.35}$$

to

$$
\begin{aligned}
&(X_0, F(l_0)(a_0)) \xrightarrow{(\phi_0, F(\phi_0 l_1)(z_0))} (X_1, F(\phi_0 l_1)(a_1)) \to \\
&\quad \cdots \to (X_{p-1}, F(\Phi_{p-2} l_{p-1})(a_{p-1})) \xrightarrow{(\Phi_{p-1}, F(\Phi_{p-1} l_p)(z_{p-1}))} (X_p, F(\Phi_{p-1} l_p)(a_p)).
\end{aligned} \tag{3.36}
$$

Consider the simplicial homotopy given by Thomason:

$$
H\colon \operatorname{hocolim} N\tilde{F} \times \Delta[1] \to N(\mathrm{Gr}(F)) \tag{3.37}
$$

that sends

$$
\begin{aligned}
&(X_0 \xrightarrow{\phi_0} X_1 \to \cdots X_{p-1} \xrightarrow{\phi_{p-1}} X_p, (l_0, a_0) \xrightarrow{(k_0, z_0)} (l_1, a_1) \to \\
&\qquad\qquad \cdots \to (l_{p-1}, a_{p-1}) \xrightarrow{(k_{p-1}, z_{p-1})} (l_p, a_p), f_i\colon [p] \to [1])
\end{aligned} \tag{3.38}
$$

where $f(0) = \cdots = f(i) = 0$ and $f(i+1) = \cdots = f(p) = 1$, to

$$
\begin{aligned}
&(Y_0, a_0) \to (Y_1, a_1) \to \cdots \to (Y_{i-1}, a_{i-1}) \xrightarrow{(\Phi_{i-1} l_{i-1}, F(\Phi_{i-1} l_i)(z_i))} \\
&\qquad (X_i, F(\Phi_{i-1} l_i)(a_i)) \to (X_{i+1}, F(\Phi_i)(l_{i+1} a_{i+1})) \to \cdots \to (X_p, F(\Phi_{p-1} l_p)(a_p))
\end{aligned} \tag{3.39}
$$

This is equivariant because each of the assignment of the point (3.38) to one of the spaces or maps appearing in (3.39) is equivariant. This finishes the proof. □

Chapter 4

Equivariant Homotopy Type of Preordered Sets

We start by stating what the theorems of the previous chapter mean in the special context of preordered sets. Remember that a preordered set is a set P with a binary relation $\leq$ which is reflexive and transitive. If G is a group, we say that P is a G-preordered set if G acts on the set P and for every $g \in G$ and $x \leq y$ in P we have $gx \leq gy$. Here, we will consider mostly finite groups and finite preordered sets.

We define the poset associated to a preordered set P. If $x \leq y$ and $y \leq x$, we write $x \sim y$. This is an equivalence relation, because it is the relation of being isomorphic when P is considered as a category. Let $[P]$ be the set of equivalence classes in P under the relation $\sim$. It is immediate to check that $[P]$ is a poset if we define $[x] \leq [y]$ if $x \leq y$. If P has a G-action, then $[P]$ has a G-action by $g[x] = [gx]$, and this makes $[P]$ a G-poset. We say $[P]$ is the G-poset *associated* to the G-preordered set P.

We note then that the associated $[P]$ without the G-action is isomorphic to the poset that would be obtained considering P as a category and taking a skeletal subcategory. It is well known (see for example [Jac89, page 27]) that a category is equivalent to any skeletal subcategory. Hence, because of Corollary 3.6, every preordered set is ordinarily homotopy equivalent to the associated poset. Therefore the study of the ordinary homotopy type of a preordered set P is all contained in the study of that of posets. However, when there is an action of G on P we see no obvious finite G-poset associated to P that has the same G-homotopy type. In this direction, we prove the following result. We remind the reader that the *barycentric subdivision* $\mathrm{sd}(P)$ of a preordered set P is the poset of chains in P, ordered by inclusion. It has a naturally defined G-action if P has one, and will be infinite whenever the preordered set has an equivalent class of size greater than one, that is, if it is not a poset.

Proposition 4.1. *Let P be a finite G-preordered set. Then* $\mathrm{sd}(P) \simeq_G P$.

Proof. Let $f\colon \mathrm{sd}(P) \to P$ be $f(x_0 \leq \cdots \leq x_n) = x_n$. Clearly f is order preserving and equivariant. For $x \in P$, we get that $f^{-1}(X_{\leq x}) = \{ x_0 \leq \cdots \leq x_n \mid x_n \leq x \}$. We can give a G_x-contraction in $\mathrm{sd}(P)$ by the maps $(x_0 \leq \cdots \leq x_n) \leq (x_0 \leq \cdots \leq x_n \leq x) \geq x$. □

4.1 The Basic Theorems for Preordered Sets

Lemma 4.2. *(Compare with Lemma 3.5) Let P and Q two G-preordered sets If $F, F'\colon P \to Q$ are equivariant order-preserving maps such that $F(x) \leq F'(x)$ for all $x \in P$, then F is G-homotopic to F'.*

An initial object x in a preordered set P seen as a category has the property that $x \leq y$ for all $y \in P$. We will call then x an *initial* element of P.

Corollary 4.3. *(Compare with Corollary 3.7) If P is a G-preordered set with an initial element fixed by G, then P is G-contractible.*

Remark 4.4. (Compare with Definition 3.9) If $\phi\colon P \to Q$ is a map of preordered sets, then with respect to the identification (2.4), if $y \in Q$, then $y\backslash\phi$ can be identified with the sub-preordered set $\{ x \in P \mid y \leq \phi(x) \}$ of P. We denote this set as $\phi^{-1}(Y_{\geq y})$

Theorem 4.5. *(Compare with Theorem 3.10) Let $\phi\colon P \to Q$ a map of preordered sets. If $\phi^{-1}(Q_{\geq y})$ is G_y-contractible for all $y \in Q$, then ϕ is a G-homotopy equivalence.*

We now generalize (1.7) of [TW91] to the case of a pre-ordered set $(X, \leq)$ with a group action. We need some notation. Define $x \prec y$ if $x \leq y$ and $x \not\sim y$ (equivalently, if $x \le y$ and $y \not\le x$). A chain

$$x_0 \prec x_1 \prec \cdots \prec x_N \tag{4.1}$$

is said to have length N. If $x \in X$, denote by $X_{\succ x}$ the set $\{\, y \in X \mid y \succ x \,\}$. We say that an element x_0 in X is minimal if $w \le x_0$ implies $w \sim x_0$. Note that the set of minimal elements is invariant under G and preserves the relation $\sim$, that is, if x is minimal and $x \sim z$, then z is minimal.

Proposition 4.6. *Let X be a G-preordered set such that there is no infinite chain of the form* (4.1) *and Y be a G-invariant subset of X that preserves the relation $\sim$. Assume that for all $x \in X - Y$, we have that $X_{\succ x}$ is G_x-contractible. Then the inclusion $Y \to X$ is a G-homotopy equivalence.*

Proof. We construct a chain of preordered sets.

$$X = X_0 \ge X_1 \ge \cdots \ge X_n = Y \tag{4.2}$$

Let X_1 be obtained from $X_0 = X$ by removing the minimal elements in $X_0 - Y$, then X_2 is obtained from X_1 removing the minimal elements of $X_1 - Y$ and so on. We reach Y in a finite number of steps because of the finite length condition. We prove $X_1 \simeq_G X_0$ using Theorem 4.5.

We note first that X_1 preserves the equivalence relation: If $x \notin X_1$ this means that x was removed at the first stage and so it is minimal in $X - Y$. Suppose x is not minimal in X. Then if $w \prec x$ for $w \in X$, given that x is minimal in $X - Y$, we must have $w \notin X - Y$, this is, $w \in Y$. Hence $x \notin X_1$ implies that either a) x is a minimal element of X or b) x has the property that $w \prec x$ implies $w \in Y$. In the first case, if $z \sim x$ then z is also minimal by the comments before the proof and cannot be in Y since Y preserves the relation and $x \notin Y$. In the second case, if $z \sim x$ and $t \prec z$, then $t \le z \le x$ and if we had $t \ge x$ we would have $t \ge z$ contradicting $t \prec z$. Hence $t \prec x$ and so $t \in Y$. So z also has the property and therefore $z \notin X_1$. This proves that the complement of X_1 preserves the relation $\sim$, hence also X_1 does.

Let $i\colon X_1 \to X$ the inclusion. Let $x \in X$ and consider $x\backslash i = \{\, z \in X_1 \mid x \le z \,\}$. We have two cases:

If $x \in X_1$, then $x\backslash i$ has x as a initial element, and it is fixed by G_x, so it is G_x-contractible, by Corollary 4.3.

If $x \notin X_1$, we now prove that

$$\{ z \in X_1 \mid x \leq z \} = X_{\succ x} \tag{4.3}$$

Let $z \in X_1$ with $z \geq x$. We cannot then have $z \sim x$, since X_1 preserves $\sim$. Hence $z \succ x$, and this proves one containtment.

Let now $z \succ x$. Then $z \geq x$ but we also need to prove $z \in X_1$. Clearly z is not minimal in X. And from the fact that $x \notin X_1$, we know that $x \notin Y$, so z does not satisfy condition b) above either. Hence $z \in X_1$, and this proves equation (4.3). Since by hypothesis $X_{\succ x}$ is G_x-contractible, we are done since Theorem 4.5 applies. $\square$

Of course we can similarly prove the dual proposition, namely

Proposition 4.7. *Let X be a G-pre-ordered set of finite length and Y be a G-invariant subset of X that preserves the relation $\sim$. Assume that for all $x \in X - Y$, we have that $X_{\prec x}$ is G_x-contractible. Then the inclusion $Y \to X$ is a G-homotopy equivalence.*

4.2 Homotopy Colimits of Preordered Sets

Let $D\colon P \to \mathbf{PreordSet}$ be a functor where P is a poset and suppose we have an action of G on D by natural transformations. Considering preordered sets as categories, and hence as simplicial sets via the nerve functor we can form $\operatorname{hocolim} D$, which *a priori* is a simplicial set. However, using Theorem 3.23, we will see in the next result that it is actually up to weak G-homotopy equivalence the nerve of a preordered set. This is because the Grothendieck construction of the functor D has as objects the points (p, x) with $p \in P$ and $x \in D(p)$ and a map from (p, x) to (p', x') if $p \leq p'$ and $D(p \leq p')(x) \leq x'$. We observe that there can be at most one map between any two objects, hence the category $\operatorname{Gr}(D) \simeq \operatorname{hocolim} D$ can be seen as a preordered set. We use now the universal property of the Grothendieck construction ([Tho79]) to give the universal property of the homotopy colimit of a functor of preordered sets:

Proposition 4.8. *The homotopy colimit of a functor* $D\colon P \to \mathbf{PreordSet}$ *has the following universal property: It is a preordered set* $\operatorname{hocolim} D$ *together with maps* $\chi_p\colon D(p) \to \operatorname{hocolim} D$*, one for each* $p \in P$ *with the property that* $\chi_p(x) \leq \chi_{p'}(D(p \leq p')(x))$ *if* $p \leq p'$, $x \in D(p)$*, and such that if there is a preordered set* Q *and a collection of maps* $\chi'_p\colon D(p) \to Q$ *with* $\chi'_p(x) \leq \chi'_{p'}(D(p \leq p')(x))$ *if* $p \leq p'$*, then there is a unique map* $\operatorname{hocolim} D \to Q$ *such that the appropriate diagrams commute.*

Proof. Define $\chi_p\colon D(p) \to \operatorname{hocolim} D$ by $\chi_p(x) = (p, x)$. If we have maps $\chi'_p\colon D(p) \to Q$ with the properties stated, define $\operatorname{hocolim} D \to Q$ by $(p, x) \mapsto \chi'_p(x)$. This is order preserving because if $(p, x) \leq (p', x')$ then $D(p \leq p')(x) \leq x'$. Applying $\chi'_{p'}$ to this last inequality we get that $\chi'_{p'}D(p \leq p')(x) \leq \chi'_{p'}x'$. Since we also know that $\chi'_p(x) \leq \chi'_{p'}(D(p \leq p')(x))$, we get $\chi'_p(x) \leq \chi'_{p'}(x')$ as we wanted. □

Hence we can take as underlying set for $\operatorname{hocolim} D$ the disjoint union $\coprod_{p\in P} D(p)$ with order relation $x \leq x'$, $x \in D(p)$, $x' \in D(p')$ if $p \leq p'$ and $D(p \leq p')(x) \leq x'$.

We now present a lemma which describes a general (finite) preordered set as a homotopy colimit of contractible spaces. This generalizes Proposition 5.1 of [WZŽ98]. Since we will work with the poset opposite to barycentric subdivision, we will use the following notation: A chain will be denoted with a bar, as in $\bar{x}$, and if a chain $\bar{x}$ includes the chain $\bar{y}$, we write $\bar{x} \supseteq \bar{y}$, hence in this situation we have $\bar{x} \leq \bar{y}$ in the opposite poset.

Lemma 4.9. *Let* X *be a finite* G*-preordered set. Let* $[X]$ *be the* G*-poset associated to* X*. Let* $P = \operatorname{sd}([X])$ *be the* G*-poset of chains of* $[X]$ *(barycentric subdivision). Let* $D\colon P^{\mathrm{op}} \to \mathbf{PreordSet}$ *the following diagram: If* $\bar{x} = ([x_0] < \cdots < [x_n]) \in P$*, then* $D(\bar{x}) = \prod_{i=0}^{n}[x_i]$ *(external product). If* $\bar{x} = ([x_0] < \cdots < [x_n]) \supseteq ([y_0] < \cdots < [y_m]) = \bar{y}$*, then* $D(\bar{x}) \to D(\bar{y})$ *is the canonical projection. Define the map* $\eta_{g,\bar{x}}$ *as*

$$\begin{aligned} D(\bar{x}) &\longrightarrow D(\bar{gx}) \\ (c_i) &\longmapsto (gc_i) \end{aligned} \tag{4.4}$$

This is an action of G *to the functor* D*, and it gives a* G*-action to* $\operatorname{hocolim} D$*. Then* $X \simeq_G \operatorname{hocolim} D$

Proof. By Corollary (3.17), we know that $\operatorname{hocolim} D$ has a G-action. By Theorem 3.23, we know that $\operatorname{hocolim} D \simeq_G \operatorname{Gr}(D)$, the Grothendieck construction associated to D. That can be considered as a preordered set Z with points $\{ (\bar{x}, (c_i)) \mid \bar{x} \in P, c_i \in [x_i] \}$ and order relation $(\bar{x}, (c_i)) \leq (\bar{y}, (d_j))$ if and only if $\bar{x} \supseteq \bar{y}$ and $\operatorname{proj}_{D(\bar{x}) \to D(\bar{y})}(c_i) \leq d_j$. The action of G on $\operatorname{hocolim} D$ corresponds to the action of G on Z given by

$$g\,(\bar{x}, (c_i)) = (g\bar{x}, (gc_i)) \tag{4.5}$$

We prove now $X \simeq_G Z$. We define a map $f\colon Z \to X$ by $f\,(\bar{x}, (c_i)) = c_0$. We check this is order preserving and equivariant. That f is equivariant is clear by equation (4.5). To prove it is order preserving, take $(\bar{x}, (c_i)) \leq (\bar{y}, (d_j))$, then $\bar{y} \subseteq \bar{x}$ in P implies that $[y_0] \geq [x_0]$ in $[X]$. Given that $c_0 \in [x_0]$ and $d_0 \in [y_0]$, we get $c_0 \leq d_0$ in X. We proceed then to apply Theorem 4.5.

Now, let $s \in X$. We want to prove that $f^{-1}(X_{\geq s})$ is G_s-contractible. We have

$$f^{-1}(X_{\geq s}) = \{ (\bar{x}, (c_i)) \in Z \mid [s] \leq [x_0] \} \tag{4.6}$$

Let us call this set L. Define a map $\phi\colon L \to L$ by

$$(\bar{x}, (c_i)) = ([x_0] < \cdots < [x_n], (c_i)) \mapsto \begin{cases} ([s] < [x_0] < \cdots < [x_n], (s, (c_i))) & [s] \neq [x_0] \\ (\bar{x}, (c_i)) & [s] = [x_0] \end{cases} \tag{4.7}$$

Then ϕ is an order preserving G_s-map and also we have $\phi(w) \leq w$ for all $w \in L$. Thus $L \simeq_{G_s} \phi(L)$. Also, since $\phi(w) \leq ([s], s)$ for all $w \in L$, $\phi(L)$ has a terminal object. Hence L is G_s-contractible, and by Theorem 4.5, f is a G-homotopy equivalence. $\square$

The spaces $D(\bar{x})$ in Lemma 4.9 are all contractible because their elements are all comparable, hence any of them is an initial element.

We now present a theorem that expresses the G-preordered set X as homotopy colimit of simpler G-preordered sets. Note that in Lemma 4.9, the diagram D was a diagram of preordered sets, but they were not G-preordered sets and the G-action on the homotopy colimit came by permuting the preordered sets among themselves.

The idea now is to group together the preordered sets in an orbit so as to obtain a group action on a single preordered set. First, observe that the G-poset $P = \mathrm{sd}([X])$ has the property that for $\bar{x}, \bar{y} \in P$, we have that $g\bar{x} \subseteq \bar{y}$ and $\bar{x} \subseteq \bar{y}$ imply that $g\bar{x} = \bar{x}$. So we can define the poset $(P/G)^{\mathrm{op}}$ by declaring $G\bar{x} \leq G\bar{y}$ if there is a $g \in G$ such that $\bar{x} \supseteq g\bar{y}$ as chains.

Theorem 4.10. *Let X, $[X]$, P and the diagram D as in the previous lemma. Consider the diagram $(P/G)^{\mathrm{op}} \to$ **PreordSet** defined as follows: $F(G\bar{x}) = \coprod_{u \in G\bar{x}} D(u) = D(\bar{x}) \uparrow_{G_{\bar{x}}}^{G}$. If $G\bar{x} \leq G\bar{y}$ in $(P/G)^{\mathrm{op}}$, we have that $\bar{x} \supseteq g\bar{y}$ for some $g \in G$. Let $h_1, \ldots, h_n$ be a collection of left coset representatives of $G_{\bar{x}}$ in G. The map $F(G\bar{x}) \to F(G\bar{y})$ is defined by the maps $D(h_i\bar{x}) \to D(h_i g\bar{y})$. Then $X \simeq_G \operatorname{hocolim} F$.*

Proof. Clearly the map $F(G\bar{x}) \to F(G\bar{x})$ associated to $G\bar{x} \leq G\bar{x}$ is the identity. Now suppose $G\bar{x} \leq G\bar{y} \leq G\bar{z}$ and $\bar{x} \supseteq g_0\bar{y}$, $\bar{y} \supseteq g_1\bar{z}$. We have that $F(G\bar{x} \leq Gg_0\bar{y})$ maps $D(\bar{x})$ to $D(g_0\bar{y})$. Let $h \in G$ such that $h^{-1}g_0 \in G_{\bar{y}}$, then $F(\bar{y} \leq g_1\bar{z})$ will map $D(g_0\bar{y})$ to $D(hg_1\bar{z})$. From $\bar{y} \supseteq g_1\bar{z}$ we get $h^{-1}g_0\bar{y} \supseteq g_1\bar{z}$, hence $\bar{y} \supseteq g_0^{-1}hg_1\bar{z}$. We deduce that $g_1\bar{z} = g_0^{-1}hg_1\bar{z}$, hence $g_0g_1\bar{z} = hg_1\bar{z}$. This proves that F is actually a diagram. We will prove $\operatorname{hocolim} F \simeq_G \operatorname{hocolim} D$ by proving that $K(F, (P/G)^{\mathrm{op}})$ as a G-bisimplicial set is G-isomorphic to $K(D, P^{\mathrm{op}})$. Now, $K(D, P^{\mathrm{op}})_n = \coprod_{\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n} D(\bar{y}_0)$ and $K(F, (P/G)^{\mathrm{op}})_n = \coprod_{G\bar{x}_0 \supseteq \cdots \supseteq G\bar{x}_n} F(G\bar{x}_0)$. We define a map $K(D, P^{\mathrm{op}})_n \xrightarrow{\phi} K(F, (P/G)^{\mathrm{op}})_n$ by sending $(\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c)$ to $(G\bar{y}_0 \supseteq \cdots \supseteq G\bar{y}_n, c)$. To define the inverse ψ, we note first that a chain $G\bar{x}_0 \supseteq \cdots \supseteq G\bar{x}_n$ determines $t_1, \ldots, t_n$, which are elements of the group G with the property that $\bar{x}_0 \supseteq t_1\bar{x}_1 \supseteq \cdots \supseteq t_n\bar{x}_n$. So, to $(G\bar{x}_0 \supseteq \cdots \supseteq G\bar{x}_n, d)$ with $d \in F(G\bar{x}_0)$ we associate $(g\bar{x}_0 \supseteq gt\bar{x}_1 \supseteq \cdots \supseteq gt_n\bar{x}_n, d)$. Then $\psi\phi(\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c) = \psi(G\bar{y}_0 \supseteq \cdots \supseteq G\bar{y}_n, c) = (\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c)$. On the other hand, $\phi\psi(G\bar{x}_0 \supseteq \cdots \supseteq G\bar{x}_n, d) = (g\bar{x}_0 \supseteq gt_1\bar{x}_1 \supseteq \cdots \supseteq gt_n\bar{x}_n, d) = (Gg\bar{x}_0 \supseteq Ggt_1\bar{x}_1 \supseteq \cdots \supseteq Ggt_n\bar{x}_n, d) = (g\bar{x}_0 \supseteq gt_1\bar{x}_1 \supseteq \cdots \supseteq gt_n\bar{x}_n, d)$. To prove ψ is simplicial, we prove it commutes with d_0: We have $\psi d_0(\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c) = \psi(\bar{y}_1 \supseteq \cdots \supseteq \bar{y}_n, D(c)) = (G\bar{y}_1 \supseteq \cdots \supseteq G\bar{y}_n, D(c))$. On the other side, $d_0\psi(\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c) = d_0(G\bar{y}_0 \supseteq \cdots \supseteq G\bar{y}_n, c) = (G\bar{y}_1 \supseteq \cdots \supseteq G\bar{y}_n, F(\bar{y}_0 \leq \bar{y}_1)c)$. But since $\bar{y}_0 \supseteq \bar{y}_1$, we can take $h = 1$ in the definition of $F(c)$. Hence $F(c) = D(c)$, and so ϕ is simplicial. Now we prove ϕ is equivariant:

$\phi(g(\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c)) = \phi(g\bar{y}_0 \supseteq \cdots \supseteq g\bar{y}_n, gc) = (Gg\bar{y}_0 \supseteq \cdots \supseteq Gg\bar{y}_n, gc) = (G\bar{y}_0 \supseteq \cdots \supseteq G\bar{y}_n, gc) = g(G\bar{y}_0 \supseteq \cdots \supseteq G\bar{y}_n, c) = g\phi(\bar{y}_0 \supseteq \cdots \supseteq \bar{y}_n, c)$

$\square$

Chapter 5

Dwyer's Space

In this chapter we define a space associated to a collection of subgroups, that generalizes the construction of the classifying space of a group.

5.1 Definition

Definition 5.1. Let G a finite group. A *collection* $\mathcal{C}$ of subgroups of G is a set of subgroups of H that is closed under conjugation.

Let $\mathcal{C}$ be a collection of subgroups of G. In [Dwy97] and [Dwy98b], W. G. Dwyer associates to $\mathcal{C}$ a space which is the homotopy colimit of the functor

$$\tilde{\beta}_{\mathcal{C}} \colon \mathbf{O}_{\mathcal{C}} \to \mathbf{GSet} \tag{5.1}$$

where $\mathbf{O}_{\mathcal{C}}$ is the category with set of objects equal to $\{G/H\}_{H\in\mathcal{C}}$ and morphism are G-maps, the category **GSet** is the category of G-sets and $\tilde{\beta}_{\mathcal{C}}$ sends G/H to G/H itself. By Theorem 1.2 of [Tho79], this space is homotopy equivalent to the nerve of the Grothendieck Construction associated to $\tilde{\beta}_{\mathcal{C}}$. This is a category which Dwyer denotes by $\mathbf{X}^{\beta}_{\mathcal{C}}$ that has as objects the pairs $(xH, G/H)$, where $H \in \mathcal{C}$, xH is a coset in the G-set G/H, and a map from $(xH, G/H)$ to $(yK, G/K)$ is a G-map $G/H \to G/K$ that sends xH to yK. The G-action is given by $g(xH, G/H) = (gxH, G/H)$. Note that the stabilizer of $(xH, G/H)$ is ${}^{x}H$.

Since the G-sets G/H are transitive, a G-map is determined by a value on a point. Hence between any two objects $(xH, G/H)$ and $(yK, G/K)$ there can be at most one map, precisely when ${}^xH \leq {}^yK$. This says that the nerve of $\mathbf{X}_{\mathcal{C}}^{\beta}$ which Dwyer denotes $X_{\mathcal{C}}^{\beta}$, is equivalent to the nerve of a pre-ordered set $P_{\mathcal{C}}$, whose elements are the cosets xH, with $H \in \mathcal{C}$ and $xH \leq yK$ if ${}^xH \leq {}^yK$.

Example 5.2. Suppose that $\mathcal{C}$ contains just the trivial group 1. Then the elements of $P_{\mathcal{C}}$ can be identified with the elements of G, and every element is comparable to each other. Hence $P_{\mathcal{C}}$ is contractible, and we also note that the action of G on it is free. Therefore, we can take $P_{\mathcal{C}}$ as a model for the space EG.

We prove now theorems that allows us to simplify the preordered set $P_{\mathcal{C}}$. For this preordered set, we have that $xH \sim yK$ if ${}^xH = {}^yK$, and so $(P_{\mathcal{C}})_{\succ xH} = \{ \, yK \mid {}^yK > {}^xH \, \}$. Since $G_{xH} = {}^xH$, we have that this stabilizer acts trivially on $(P_{\mathcal{C}})_{\succ xH}$.

Theorem 5.3. *Let $\mathcal{C}$ be a G-poset of subgroups of G ordered by inclusion and with G acting by conjugation. Let $l\colon \mathcal{C} \to \mathcal{C}$ be a G-poset map such that $H \subset l(H)$ for all $H \in \mathcal{C}$. Then l induces a G-preordered set map $P_l\colon P_{\mathcal{C}} \to P_{\mathcal{C}}$ by $P_l(xH) = xl(H)$, and we have that $P_{\mathcal{C}} \simeq_G \operatorname{im} P_l \simeq_G P_{\operatorname{im} l}$.*

Proof. We show first that P_l is well-defined: if $xH = yH$ then $y^{-1}x \in H \subset l(H)$, hence $xl(H) = yl(H)$. Now we prove it preserves the order of $P_{\mathcal{C}}$: let $xH \leq yK$, then ${}^xH \subset {}^yK$, hence $l({}^xH) \subset l({}^yK)$, hence ${}^xl(H) \subset {}^yl(K)$, so $xl(H) \leq yl(K)$, but this is precisely $P_l(xH) \leq P_l(yK)$. To prove P_l preserves the action of G, consider $P_l(g(xH)) = P_l(gxH) = gxl(H) = g(xl(H)) = gP_l(xH)$.

To prove the final statement, just note that $xH \leq P_l(xH)$ for any $xH \in P_{\mathcal{C}}$. The theorem then follows from Lemma 4.2. $\square$

We show some examples

1. Let $\mathcal{C}$ a collection of subgroups (closed under intersection) and $l(K) =$ intersection of maximal elements of $\mathcal{C}$ containing K. This proves $P_{\mathcal{C}}$ to be G-homotopy equivalent to $P_{\mathcal{C}'}$, where $\mathcal{C}'$ contains the subgroups that are intersection of maximal elements of $\mathcal{C}$.

2. $\mathcal{C} = \mathcal{S}_p(G)$, the poset of nontrivial p-subgroups of G, Q a normal p-subgroup and $l(K) = KQ$. Here $\operatorname{im} l$ consists of the subgroups in $\mathcal{C}$ containing Q.

Proposition 5.4. *Let $\mathcal{C}_0$ be a set of representatives of the conjugacy classes of $\mathcal{C}$. Then $P_{\mathcal{C}_0}$ is G-homotopy equivalent to $P_{\mathcal{C}}$.*

Proof. For each $K \in \mathcal{C}$, denote by $\hat{K} \in \mathcal{C}_0$ the chosen representative and fix $g_K \in G$ such that $\hat{K} = {}^{g_K}K$. If $K \in \mathcal{C}_0$, choose $g_K = 1$. Then we have an isomorphism of G-sets $c_{g_K} \colon G/K \to G/\hat{K}$ given by $xK \mapsto xg_K^{-1}\hat{K}$. Define then a map $f \colon P_{\mathcal{C}} \to P_{\mathcal{C}_0}$ by sending $xK \to c_{g_K}(xK)$. Since we have $c_{g_K}(xK) \leq xK \leq yL \leq c_{g_L}(yL)$ for $xK \leq yL$, we have that f is order preserving. We have also that $gf(xK) = g(xg_K^{-1}\hat{K}) = gxg_K^{-1}\hat{K} = f(gxK)$ and so f is equivariant. On the other hand, the inclusion $i \colon P_{\mathcal{C}_0} \to P_{\mathcal{C}}$ is of course order preserving and equivariant. Then we check: Let $xK \in P_{\mathcal{C}}$, then $if(xK) = i(xg_K^{-1}\hat{K}) = xg_K^{-1}\hat{K} \geq xK$, and so $if \simeq 1_{P_{\mathcal{C}}}$ by Lemma 3.5. Since $fi = 1_{P_{\mathcal{C}_0}}$, we get $P_{\mathcal{C}_0} \simeq P_{\mathcal{C}}$ □

Proposition 5.5. *Let $\mathcal{C}$ be a collection of subgroups of G closed under conjugation and let $\mathcal{C}_0$ be a set of representatives of the conjugacy classes of $\mathcal{C}$. Then the G-posets $\mathcal{C}$ (where G acts by conjugation) and $[P_{\mathcal{C}_0}]$ are G-isomorphic.*

Proof. Fix a set map $\phi \colon \mathcal{C} \to \mathcal{C}_0$ such that for each $H \in \mathcal{C}$ we have that $\phi(H)$ is the chosen representative of the conjugacy class of H. Define maps $\Lambda \colon [P_{\mathcal{C}_0}] \to \mathcal{C}$ and $\Gamma \colon \mathcal{C} \to [P_{\mathcal{C}_0}]$ by $\Lambda([xH]) = {}^{x}H$ and $\Gamma(H) = [a\phi(H)]$ where $H = {}^{a}\phi(H)$. It is routine to prove they are well-defined order-preserving equivariant bijections. □

Proposition 5.6. *If $H \in P_{\mathcal{C}_0}$, then the equivalence class of H in $P_{\mathcal{C}_0}$ with respect to the relation $\sim$, which we denoted by $[H]$, is isomorphic as an $N(H)/H$-preordered set to* $\mathrm{E}(N(H)/H)$.

Proof. If $yH' \sim H$, then ${}^{y}H' = H$. Since H is the only element of its conjugacy class in $\mathcal{C}_0$, we must have that $H' = H$ and so $y \in N(H)/H$. Hence the elements of $[H]$ can be identified with the cosets of H in $N(H)$. By Example 5.2 we can identify then $[H]$ with $\mathrm{E}(N(H)/H)$. □

5.2 The case $\mathcal{C} = \mathcal{S}_p(G)$

In this section we prove a result analogous to one previously proved by Quillen ([Qui78, Proposition 2.6]), expressing the poset of $\mathcal{S}_p(G_1 \times G_2)$ as the join of $\mathcal{S}_p(G_1)$ and $\mathcal{S}_p(G_2)$. To this end, first we prove a lemma which identifies the join of two preordered sets in a certain way. The *join* of two preordered sets X and Y is the preordered set that has as underlying set the disjoint union of X and Y, the order relation is given by $w \le z$ if either one of the three following statements is true

1. $w, z \in X$ and $w \le z$ in X
2. $w, z \in Y$ and $w \le z$ in Y
3. $w \in X$ and $z \in Y$

Lemma 5.7. *Let X and Y be G-preordered sets. Let $Z = Z(X,Y)$ be the following preordered set: The underlying set of Z is $X \sqcup Y \sqcup X \times Y$ and the order relation $x \le (x,y)$, $y \le (x,y)$, and the usual order in X, Y and $X \times Y$. Then Z has a natural G-action defined, and we have $Z \simeq_G X * Y$*

Proof. We will identify X, Y and $X \times Y$ with their images in Z under inclusion. We now prove $Z \simeq X * Y$ using Quillen's theorem. Consider the map $f \colon Z \to X * Y$ that sends x to x, y to y and (x,y) to y. Let $x_0 \in X$, then $f^{-1}\left((X*Y)_{\le x_0}\right) = X_{\le x_0}$, which is G_{x_0}-contractible. Now, for $y_0 \in Y$, we have

$$f^{-1}\left((X*Y)_{\le y_0}\right) = X \cup Y_{\le y_0} \cup \{\, (x,y) \mid y \le y_0 \,\} \tag{5.2}$$

We will denote $L = f^{-1}\left((X*Y)_{\le y_0}\right)$. We define the map $\phi \colon L \to L$ that sends $x \in X$ to (x, y_0), $y \in Y_{\le y_0}$ to y_0 and $(x,y) \in \{\, (x,y) \mid y \le y_0 \,\}$ to (x, y_0). Then ϕ is order-preserving and G_{y_0}-equivariant, and also we have $\phi(w) \ge w$ for all $w \in L$. Thus $L \simeq \phi(L)$. Also, since $\phi(w) \ge y_0$ for all $w \in L$, $\phi(L)$ has a terminal object. Hence $L = f^{-1}\left((X*Y)_{\le y_0}\right)$ is contractible, and by Theorem 3.10, f is G-a homotopy equivalence. $\square$

Remark 5.8. We will use this lemma in the following way: Suppose that X_i is a G_i-preordered set for $i = 1, 2$. Then if $G = G_1 \times G_2$, we can give a natural G-action to X, Y, $X \times Y$ and $X * Y$ (hence to Z), and with that action, we have $Z \simeq_G X * Y$.

Lemma 5.9. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be collections of subgroups of G_1, G_2 respectively. Then*

$$P_{(\mathcal{C}_1\cup\{1\})\times(\mathcal{C}_2\cup\{1\})-(1,1)} \cong_{G_1\times G_2} P_{\mathcal{C}_1} * P_{\mathcal{C}_2} \tag{5.3}$$

Proof. Let Z be constructed from $P_{\mathcal{C}_1}$, $P_{\mathcal{C}_2}$ as in Proposition 5.7. From there, we know $Z \simeq_{G_1\times G_2} P_{\mathcal{C}_1} * P_{\mathcal{C}_2}$. We will prove then equivariant homotopy equivalence of the left side of (5.3) with Z. Consider the map $f\colon P_{(\mathcal{C}_1\cup\{1\})\times(\mathcal{C}_2\cup\{1\})-(1,1)} \to Z$ defined by

$$(xH, yK) \mapsto \begin{cases} yK & H = 1 \\ xH & K = 1 \\ (xH, yK) & H \neq 1 \text{ and } K \neq 1 \end{cases} \tag{5.4}$$

and the map $g\colon Z \to P_{(\mathcal{C}_1\cup\{1\})\times(\mathcal{C}_2\cup\{1\})-(1,1)}$ defined by

$$\begin{aligned} xH &\mapsto (xH, 1) \\ yK &\mapsto (1, yK) \\ (xH, yK) &\mapsto (xH, yK) \end{aligned}$$

We have that f and g are order preserving, equivariant and inverses to each other, hence they give an isomorphism of preordered sets $\square$

Proposition 5.10. *We have that*

$$P_{\mathcal{S}_p(G_1\times G_2)} \simeq_{G_1\times G_2} P_{\mathcal{S}_p(G_1)} * P_{\mathcal{S}_p(G_2)} \tag{5.5}$$

Proof. Let $\mathcal{C} = \mathcal{S}_p(G_1 \times G_2)$. Take $l(K) = \mathrm{pr}_1(K) \times \mathrm{pr}_2(K)$. This satisfies the hypothesis of Theorem 5.3 and so we have that $P_{\mathcal{C}} \simeq_{G_1\times G_2} P_{\mathcal{C}'}$, where $\mathcal{C}' = \operatorname{im} l =$ all subgroups of G of the form $H \times K$ with $H \in \mathcal{S}_p(G_1) \cup \{1\}$, $K \in \mathcal{S}_p(G_2) \cup \{1\}$, and not both 1. From Lemma 5.9, we obtain that $P_{C'} \simeq_{G_1\times G_2} P_{S_p(G_1)} * P_{S_p(G_2)}$ $\square$

5.3 Connected Components of $P_{\mathcal{C}}$

Proposition 5.11. *Let X and Y be G-homotopy equivalent topological spaces by maps $\phi\colon X \to Y$ and $\psi\colon Y \to X$. Let $x \in X$, and $C(x)$ be the connected component of X containing x. Then $G_{C(x)} = G_{C(\phi(x))}$.*

Proof. By standard topological arguments (see [Spa66, page 49]), we know that ϕ and ψ determine a bijection of the connected components of X and Y. Now let $g \in G_{C(x)}$. We want to prove g stabilizes the component of $\phi(x)$ in Y. Let $y \in C(\phi(x))$. Since the action of g is a continuous map, we have that gy is in the same component as $g\phi(x)$, this is $C(\phi(gx))$. Since $gx \in C(x)$, we know $C(\phi(gx)) = C(\phi(x))$. Hence $gy \in C(\phi(x))$. This proves $G_{C(x)} \subseteq G_{C(\phi(x))}$. Applying the same argument to ψ we obtain $G_{C(\phi(x))} \subseteq G_{C(\psi\phi(x))}$. But $\psi\phi(x)$ is in the same component as x, finishing the proof. $\square$

For the case of the poset $\mathcal{S}_p(G)$, remember that Quillen proved in ([Qui78, Corollary 5.3]) that the stabilizer of the component containing the Sylow subgroup P was generated by $N_G(H)$ for $1 < H \subseteq P$.

Definition 5.12. Let H and K be subgroups of G. We define the *transporter* from H to K in G, $T(H, K)$, as the set

$$T(H, K) = \{\, g \in G \mid {}^{g}H \leq K \,\} = \{\, g \in G \mid H \leq K^{g} \,\}$$

Theorem 5.13. *Let $\mathcal{C}$ be a set of nontrivial subgroups of G. Let L be a maximal element in $\mathcal{C}$. Then the stabilizer of the connected component of $P_{\mathcal{C}}$ containing L is the subgroup generated by all $T(H, K)$ for $1 \leq H \leq K \leq L$, $H, K \in \mathcal{C}$.*

Proof. Let T be the subgroup generated by all $T(H, K)$ for $1 \leq H \leq K \leq L$, $H, K \in \mathcal{C}$. Let $g \in T(H, K)$ and we prove that it stabilizes the component of L, which we will denote by $C(L)$. Let xS lie in such a component. Since $H \leq K \leq L$, we know that H and K (as cosets) are also in that component. Hence gxS is in the same component as gH. But $gH \leq K$, hence gH is in the component of K, which is the component of L. This proves that T is contained in the stabilizer of the component.

To prove the other containment we will first observe that if $xS \in C(L)$, then $x \in T$. Let $xS \in C(L)$ Then there is a sequence $L = H_0, a_1H_1, \ldots, a_nH_n = xS$ so that each term in the sequence is comparable to the next as elements of $P_{\mathcal{C}}$. We prove our assertion by induction on n. If $n = 1$, so that $xS \leq L$ (the case $xS \geq L$ is similar), then $x \in T(S, L) \leq T$. This proves the case $n = 1$. Now for the general case, if $a_{n-1}H_{n-1} \leq a_n H_n$ (the case $a_{n-1}H_{n-1} \geq a_n H_n$ is similar), we have that $a_n^{-1}a_{n-1} \in T(H_{n-1}, H_n) \subset T$. Since $a_{n-1} \in T$ by induction hypothesis, we get that $a_n \in T$, proving our observation. Now let $g \in G_{C(L)}$. Since $gL \in C(L)$, we get $g \in T$. Hence $G_{C(L)} \subset T$. $\square$

It is interesting to observe that for $\mathcal{C} = \mathcal{S}_p(G)$, the stabilizer of the component containing a Sylow p-subgroup P is the same in both $\mathcal{C}$ and $P_{\mathcal{C}}$ even though they are not G-homotopy equivalent in general. To prove this, it is enough to show that if $g \in T(H, K)$ with $H \leq K \leq P$, then g is in the subgroup generated by the normalizer of the proper p-subgroups of G. But this is consequence of Alperin's Fusion Theorem ([Alp67, page 229]).

Chapter 6

Calculations

6.1 The Chain Complex of a Join

Our goal in this section is to describe the chain complex of the join of two preordered sets X and Y, that is $\mathcal{C}(X * Y)$, in terms of the chain complexes of X and Y.

We define a boundary in the total complex of the tensor product of two complexes, (see ([Mun84, page 341, exercise 3]). Let $\mathcal{C}$, $\mathcal{D}$ be chain complexes, then if $x \in C_i$, $y \in D_j$ with $i + j = n$, let

$$\partial(x \otimes y) = -\partial x \otimes y + (-1)^i x \otimes \partial y \tag{6.1}$$

This is a nonstandard definition, but the resulting chain complex is isomorphic to the usual tensor product by the chain map that sends $x \otimes y$ to $(-1)^i x \otimes y$, where $x \in C_i$.

Note the following formula in $\mathcal{C}(X * Y)$:

$$\partial(x_0 \leq \cdots \leq x_i \leq y_0 \leq \cdots y_{n-i-1}) = \partial\overline{x} \leq \overline{y} + (-1)^{i+1}\overline{x} \leq \partial\overline{y} \tag{6.2}$$

with $\overline{x} = (x_0 \leq \cdots \leq x_i)$, $\overline{y} = (y_0 \leq \cdots \leq y_{n-i-1})$. If the chain $\overline{x}$ consists only of the point x_0, then we interpret $\partial\overline{x} \leq \overline{y}$ as $\overline{y}$.

We define a chain map $f \colon \mathcal{C}(X) \otimes \mathcal{C}(Y) \to \mathcal{C}(X) \oplus \mathcal{C}(Y)$ as follows: Consider the maps

$$C_n(X) \otimes C_0(Y) \xrightarrow{1 \otimes \varepsilon} C_n(X) \otimes k \to C_n(X) \xrightarrow{(-1)^{n+1}} C_n(X) \tag{6.3}$$

$$C_0(X) \otimes C_n(Y) \xrightarrow{\varepsilon \otimes 1} k \otimes C_n(Y) \to C_n(Y) \tag{6.4}$$

Taking the direct sum of these two composites, we get a map

$$(C_n(X) \otimes C_0(Y)) \oplus (C_0(X) \otimes C_n(Y)) \to C_n(X) \oplus C_n(Y)$$

and we extend this to a map $f\colon \mathcal{C}(X) \otimes \mathcal{C}(Y) \to \mathcal{C}(X) \oplus \mathcal{C}(Y)$ by defining it equal to zero in all the other summands of $\mathcal{C}(X) \otimes \mathcal{C}(Y)$. In zero dimension, we have $f_0\colon C_0(X) \otimes C_0(Y) \to C_0(X) \oplus C_0(Y)$ defined by $x \otimes y \mapsto (-x, y)$, where $x \in X$ and $y \in Y$, that is the points x, y are basic elements of $C_0(X)$ and $C_0(Y)$ respectively. It is a routine calculation to check that f is actually a chain map (using the defintion of boundary (6.1)).

Proposition 6.1. *The chain complex* $\mathcal{C}(X * Y)$ *is isomorphic to* $\operatorname{cone}(f)$*, where* f *is the chain map just defined.*

Proof. Remember that if $f\colon \mathcal{C} \to \mathcal{D}$ is a chain map, $\operatorname{cone}(f)$ is a chain complex with

$$\operatorname{cone}(f)_n = C_{n-1} \oplus D_n, \qquad \partial(c, d) = (-\partial c, \partial d + fc) \tag{6.5}$$

To prove the claimed isomorphism, consider the map Φ which in dimension n is

$$C_n(X * Y) \xrightarrow{\Phi_n} (\mathcal{C}(X) \otimes \mathcal{C}(Y))_{n-1} \oplus C_n(X) \oplus C_n(Y) \tag{6.6}$$

given by

$$\begin{aligned} \overline{x} = (x_0 \leq \cdots \leq x_n) &\mapsto (0, \overline{x}, 0) \\ \overline{y} = (y_0 \leq \cdots \leq y_n) &\mapsto (0, 0, \overline{y}) \\ \overline{x} \leq \overline{y} = (x_0 \leq \cdots \leq x_i \leq y_0 \leq \cdots y_{n-i-1}) &\mapsto (\overline{x} \otimes \overline{y}, 0, 0) \end{aligned}$$

Again it is routine to prove that Φ is a chain map. Since each Φ_n is an isomorphism, we get that Φ is an isomorphism of chain complexes. □

6.2 An Example

In this example, we will take $\mathcal{C} = \mathcal{S}_p(G)$ for some prime p and consider the Sylow p-subgroup $H_n^G(|P_{\mathcal{C}}|; U)_p$ of the equivariant homology, where U is a G-module. There are two spectral sequences ([Bro82, page 172]) converging to this homology, one of which is the *isotropy spectral sequence*, whose E^1 page is

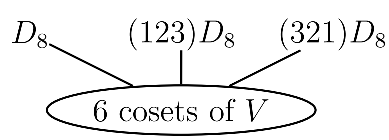
$$E^1_{rs} = \bigoplus_{\sigma \in [G \backslash N(P_{\mathcal{C}})_r]} H_s(G_\sigma, U)_p \tag{6.7}$$

Here the sum is taken over representatives of the orbits of the r-simplices and we need only consider the non-degenerate ones. This page consists of a lot of sequences which are shown in [Web91, Dwy98b] under the hypothesis $(P_{\mathcal{C}})^H$ is contractible for all $H \in \mathcal{S}_p(G)$ to be split and have homology $H_s(G, U)_p$ in degree 0. Let us call the augmented sequences

$$\cdots \to \bigoplus_{\sigma \in [G \backslash N(P_{\mathcal{C}})_1]} H_s(G_\sigma, U)_p \to \bigoplus_{\sigma \in [G \backslash N(P_{\mathcal{C}})_0]} H_s(G_\sigma, U)_p \to H_s(G, U)_p \to 0 \tag{6.8}$$

isotropy sequences. There are also similarly constructed sequences for cohomology.

We want to consider the case when $G = S_4$ and $\mathcal{C} = \mathcal{S}_p(G)$ with $p = 2$. We will apply the previous results to obtain an expression that relates the 2-part of the cohomology of G with that of some of its subgroups. By the first example after Theorem 5.3 and Proposition 5.4 we obtain that $P_{\mathcal{C}}$ is G-homotopy equivalent to $P_{\mathcal{C}_0}$, where $\mathcal{C}_0$ contains only the cosets of one copy of D_8 and the cosets of one copy the Klein's group V. Since V is contained in D_8 and V is normal, we have that $Z = P_{\mathcal{C}_0}$ can be represented by the following diagram



and the condition ${}^xV \subset {}^yD_8$ for all $x, y \in S_4$ means that in the preordered set Z every coset of V is less than every coset of D_8. Hence we have that Z can be written as the join of $X = P_{\{V\}}$ and $Y = P_{\{D_8\}}$, and so by section 6.1 the chain complex of Z can be written in terms of $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ as a cone.

Let $k = \mathbb{F}_2$ be the field of two elements. We have that X, which is the preordered set of cosets of V on S_4, can be seen as $\mathrm{E}(S_4/V)$ inflated up to S_4, since $V \lhd G$, and

$xV \leq yV$ for any pair of elements in X (see Example 5.2), hence $\mathcal{C}(X)$ is the bar resolution of k as $S_4/V \cong S_3$-module and inflated to S_4 (compare [Bro82, page 18]). We want now to find a simpler chain complex which is chain homotopy equivalent to $\mathcal{C}(X)$.

Every projective resolution of k as a kS_3-module is chain homotopy equivalent to a minimal resolution, which is given by

$$\cdots \longrightarrow k\uparrow_{C_3}^{S_3} \xrightarrow{1+(12)} k\uparrow_{C_3}^{S_3} \xrightarrow{1+(12)} k\uparrow_{C_3}^{S_3} \longrightarrow k \longrightarrow 0 \tag{6.9}$$

This is the standard resolution for k as a module over the cyclic group S_3/C_3, then inflated to S_3. The modules are still projective after inflation since they are projective when restricted to a Sylow 2-subgroup.

A concrete representation of $k\uparrow_{C_3}^{S_3}$ is $k(1+(123)+(132)) \oplus k((12)+(13)+(23))$. Identifying S_3 with S_4/V by the isomorphism given by the composition $S_3 \hookrightarrow S_4 \to S_4/V$ we may identify $k\uparrow_{C_3}^{S_3}$ with $k\uparrow_{A_4/V}^{S_4/V} = k(V+(123)V+(132)V) \oplus k((12)V + (13)V + (23)V)$. The maps between the modules are still $1+(12)$. After inflating to S_4, the modules become isomorphic to $k\uparrow_{A_4}^{S_4} = kA_4 \oplus k(12)A_4$ and the map will be $1+(12)$. Therefore, if we inflate the resolution (6.9) we obtain a chain complex $\mathcal{P}_1$ of S_4-modules that is chain homotopy equivalent to $\mathcal{C}(X)$ since before inflation the projective resolutions were chain homotopy equivalent. Each term in $\mathcal{P}_1$ is isomorphic to $k\uparrow_{A_4}^{S_4}$ and the map between them is always $1+(12)$.

Now $\mathcal{C}(Y)$ is chain homotopy equivalent to a chain complex $\mathcal{P}_2$ concentrated in dimension 0 as $k\uparrow_{D_8}^{S_4}$ since Y consists solely of three points (the three cosets of D_8) with no order relations among them. In order to make future calculations easier, choose a copy of D_8 such that $(12) \in D_8$.

By the previous section, $\mathcal{C}(Z)$ is isomorphic to $\mathrm{cone}(f)$ for a certain map $f\colon \mathcal{C}(X) \otimes \mathcal{C}(Y) \to \mathcal{C}(X) \oplus \mathcal{C}(Y)$. Replacing $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ by the chain homotopy equivalent complexes $\mathcal{P}_1$ and $\mathcal{P}_2$ we obtain that $\mathcal{C}(Z)$ will be chain homotopy equivalent to $\mathrm{cone}(g)$, where $g\colon \mathcal{P}_1 \otimes \mathcal{P}_2 \to \mathcal{P}_1 \oplus \mathcal{P}_2$ is defined analogously to the way f was.

We have that $\mathcal{P}_1 \otimes \mathcal{P}_2$ has $k\uparrow_{A_4}^{S_4} \otimes k\uparrow_{D_8}^{S_4}$ on each dimension. The boundary of this chain complex is $(1+(12)) \otimes 1$ on each dimension. Considering the map of G-sets $S_4/V \to S_4/A_4 \times S_4/D_8$ given by $xV \mapsto (xA_4, xD_8)$, we observe it is injective since if

xV and yV have the same image we obtain that $xA_4 = yA_4$ and $xD_8 = yD_8$, hence $y^{-1}x \in A_4 \cap D_8 = V$, so $xV = yV$. Since both G-sets have six elements, we conclude that such map is an isomorphism. Hence we get that $\mathcal{P}_1 \otimes \mathcal{P}_2$ is isomorphic to a chain complex $\mathcal{C}$ that has $k\uparrow_V^{S_4}$ in each dimension and we see that the boundary corresponds to $1 + (12)$ by commutativity of the following diagram

$$\begin{tikzcd}
k\uparrow_{A_4}^{S_4} \otimes k\uparrow_{D_8}^{S_4} \arrow[r] \arrow[d, "(1+(12))\otimes 1"'] & k\uparrow_V^{S_4} \arrow[d, "1+(12)"] \\
k\uparrow_{A_4}^{S_4} \otimes k\uparrow_{D_8}^{S_4} \arrow[r] & k\uparrow_V^{S_4}
\end{tikzcd}$$

(We needed here that $(12) \in D_8$). We will use also g to denote the corresponding map $g\colon \mathcal{C} \to \mathcal{P}_1 \oplus \mathcal{P}_2$. There is a short exact sequence of complexes

$$0 \longrightarrow (\mathcal{P}_1 \oplus \mathcal{P}_2) \longrightarrow \operatorname{cone} g \longrightarrow \mathcal{C}[-1] \longrightarrow 0 \tag{6.10}$$

(see [Wei94, page 19]), where we use $\mathcal{C}[-1]$ to denote the chain complex with $\mathcal{C}[-1]_n = C_{n-1}$. The low degree terms of this sequence of complexes are shown in Figure 6.1. Since from the definition of the cone this sequence of complexes splits on each dimen-

$$\begin{tikzcd}
0 \arrow[r] & k\uparrow_{A_4}^{S_4} \oplus k\uparrow_{D_8}^{S_4} \arrow[r] & \operatorname{cone}(g)_0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & k\uparrow_{A_4}^{S_4} \arrow[r] \arrow[u, "(1+(12){,}0)"] & \operatorname{cone}(g)_1 \arrow[r] \arrow[u] & k\uparrow_V^{S_4} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & k\uparrow_{A_4}^{S_4} \arrow[r] \arrow[u, "1+(12)"] & \operatorname{cone}(g)_2 \arrow[r] \arrow[u] & k\uparrow_V^{S_4} \arrow[r] \arrow[u, "1+(12)"] & 0 \\
& \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Figure 6.1:

sion as a sequence of kS_4-modules (although not a sequence of complexes), we get also a short exact sequence of complexes after applying the functor $\operatorname{Tor}_n^{kS_4}(-, k)$. We apply Shapiro's Lemma in order to identify $\mathrm{Tor}_n^{kS_4}(k\uparrow_V^{S_4}, k)$ with $H_n(V,k)$, etc. Low degree terms of the long exact sequence in homology of these short exact sequence of complexes are shown in Figure 6.2. To identify the terms in the middle column we use

$$\begin{tikzcd}
H_n(A_4)/(1+(12)) \oplus H_n(D_8) \arrow[r] & H_n(S_4) \arrow[r] & 0 \\
* \arrow[r] & 0 \arrow[r] & H_n(V)/(1+(12)) \arrow[ull]
\end{tikzcd}$$

Figure 6.2:

our knowledge that $\mathrm{Tor}_n^{kS_4}(\mathrm{cone}(g), k)$ is the isotropy sequence of S_4. By [Web87b, Theorem 3.3], such sequence is acyclic with 0-homology equal to $H_n(S_4)$.

One can prove the following facts:

$$\begin{aligned} H_n(A_4)/(1+(12)) &\cong H_n(V)_{S_3} \cong H_0(S_3, H_n(V)) \\ H_n(V)/(1+(12)) &\cong H_n(V)_{C_2} \cong H_0(C_2, H_n(V)) \end{aligned} \tag{6.11}$$

On the other hand, we have for any S_3-module M, the following:

$$H_i(S_3, M) = H_i(C_2, M) + \frac{1}{2}[H_i(C_3, M) - H_i(1, M)] \tag{6.12}$$

in the Grothendieck group of finitely generated abelian groups with relations given by direct sum decompositions. This comes from $u_{S_3} = u_{C_2} + \frac{1}{2}[u_{C_3} - u_1]$, see [Web87a, p. 159]. Putting $i = 0$ and $M = H_n(V)$, we get

$$H_n(V)_{S_3} = H_n(V)_{C_2} + \frac{1}{2}[H_n(V)_{C_3} - H_n(V)] \tag{6.13}$$

Now, from the sequence in Figure 6.2, and using the equations from 6.11, we get the following equation in the Grothendieck group

$$H_n(S_4) = H_n(D_8) + H_n(V)_{S_3} - H_n(V)_{C_2} \tag{6.14}$$

and from Equation 6.13, we get that $H_n(V)_{S_3} - H_n(V)_{C_2} = \frac{1}{2}[H_n(V)_{C_3} - H_n(V)]$. But by Swan's theorem, we know that $H_n(V)_{C_3} \cong H_n(A_4)$. So we finally arrive to

$$H_n(S_4) = H_n(D_8) + \frac{1}{2}[H_n(A_4) - H_n(V)] \tag{6.15}$$

This is the same as the formula which was obtained by Webb in [Web87a]. However the formula there had a purely combinatorial origin, and part of our goal here is to provide a geometric setting in which the coefficients may be interpreted. It does indeed appear that the formula arises in this way through consideration of the topological information in Dwyer's complex $X^{\beta}_{\mathcal{C}}$.

6.3 A Spectral Sequence Approach

We now describe a new spectral sequence associated to the space $X^{\mathcal{C}}_{\beta}$ converging to the (co)homology of G. It provides relationships between the (co)homology of G and that of its subgroups. We start by recalling the Leray spectral sequence of a map of simplicial sets. If $f\colon X \to B$ is a map of G-simplicial sets, the Leray spectral sequence of f ([Dwy98a, page 39]) is the homology spectral sequence associated to the filtration

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \tag{6.16}$$

where $X^i = f^{-1}(\mathrm{sk}_i B)$. Consider the map $f\colon P_{\mathcal{C}} \to \mathcal{C}$ given by $f(xH) = {}^{x}H$. It is equivariant and so induces a G-simplicial set map $N(f)\colon N(P_{\mathcal{C}}) \to N(\mathcal{C})$. Letting $X = N(P_{\mathcal{C}})$ and $B = N(\mathcal{C})$, we get that X^i can be identified with the set of chains of elements of $P_{\mathcal{C}}$ such that at most i morphisms are nonisomorphisms. Let $\mathfrak{C}^i$ be the chain complex of X^i, so that $\{\mathfrak{C}^i\}$ gives a filtration of $\mathfrak{C}$, the chain complex of X. We have then the situation of Figure 6.3

We now depart from what is usually done with the Leray spectral sequence. Applying the functor $\mathrm{Tor}_r(-,k)$ to this filtration gives a filtration $\{\mathrm{Tor}_r(\mathfrak{C}^i,k)\}$ of $\mathrm{Tor}(\mathfrak{C},k)$. Note that $C_n(X^i)$ is a direct summand of $C_n(X^{i+1})$ as $\mathbb{Z}G$-modules (although $\mathfrak{C}^i$ is not a direct summand in general of $\mathfrak{C}^{i+1}$ as complexes of $\mathbb{Z}G$-modules). This is because as a G-set X^{i+1} is the disjoint union of X^i and some other G-set. Hence when we apply $\mathrm{Tor}_r(-,k)$ to this diagram the morphisms $\mathrm{Tor}_r(C_n(X^i),k) \to \mathrm{Tor}_r(C_n(X^{i+1}),k)$ are (split) monomorphisms, and so $\mathrm{Tor}_r(\mathfrak{C}^i,k) \to \mathrm{Tor}_r(\mathfrak{C}^{i+1},k)$ is a monomorphism of chain complexes. Because of the splitting in each degree, we also get that $\mathrm{Tor}_r(\mathfrak{C}^i,k)/\mathrm{Tor}_r(\mathfrak{C}^{i-1},k)$ is isomorphic to $\mathrm{Tor}_r(\mathfrak{C}^i/\mathfrak{C}^{i-1},k)$. Hence

$$\begin{tikzcd}[column sep=small]
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\cdots \subset & C_n(X^i) & \subset & C_n(X^{i+1}) & \subset \cdots \subset & \mathcal{C}^i \\
\cdots \subset & C_{n-1}(X^i) \arrow[from=u] \arrow[d] & \subset & C_{n-1}(X^{i+1}) \arrow[from=u] \arrow[d] & \subset \cdots \subset & \mathcal{C}^{i+1} \\
 & \vdots & & \vdots & &
\end{tikzcd}$$

Figure 6.3:

the spectral sequence associated to the filtration $\{\mathrm{Tor}(\mathcal{C}_i, k)\}$ has E^0 page $E^0_{mn} = \mathrm{Tor}_r(C^m_{m+n}/C^{m-1}_{m+n}, k)$. Consider the bottom sequence of the E^1 page:

$$E^1_{0,0} \xleftarrow{d^1} E^1_{1,0} \xleftarrow{d^1} E^1_{2,0} \tag{6.17}$$

Since we have a first quadrant spectral sequence, we have that $E^\infty_{0,0} = H_r(G, k)$ is the cokernel of d^1. Also, the sequence (6.17) is exact at the term $E^1_{1,0}$ because its homology at that term is an image of the first homology of the isotropy sequence, which is zero. This proves

Theorem 6.2. *The sequence*

$$0 \leftarrow H_r(G, k) \leftarrow E^1_{0,0} \xleftarrow{d^1} E^1_{1,0} \xleftarrow{d^1} E^1_{2,0} \tag{6.18}$$

is exact.

We also mention the following consequences of the fact that the spectral sequence converges to a chain complex concentrated in degree 0.

Proposition 6.3. *When the dimension of the poset $\mathcal{C}$ is 1, we have a short exact sequence*

$$0 \leftarrow H_r(G, k) \leftarrow E^1_{0,0} \xleftarrow{d^1} E^1_{1,0} \leftarrow 0 \tag{6.19}$$

Proposition 6.4. *When the dimension of the poset $\mathcal{C}$ is 2, the E^1 consists of maps which fit together in a long exact sequence*

$$\begin{array}{c} \xleftarrow{(d^2)^{-1}} E^1_{0,s} \longleftarrow E^1_{1,s} \longleftarrow E^1_{2,s} \\ \\ \xleftarrow{(d^2)^{-1}} E^1_{0,s-1} \longleftarrow E^1_{1,s-1} \longleftarrow E^1_{2,s-1} \\ \\ \vdots \\ \\ 0 \longleftarrow H_r(G,k) \longleftarrow E^1_{0,0} \longleftarrow E^1_{1,0} \longleftarrow E^1_{2,0} \end{array} \tag{6.20}$$

In order to use these results we should be able to identify the complexes $\mathcal{C}^m/\mathcal{C}^{m-1}$. Remember the notation of Proposition 5.5. We have

Proposition 6.5. *We have the following isomorphism of G-chain complexes*

$$\begin{aligned} \mathcal{C}^m/\mathcal{C}^{m-1} &\cong \bigoplus_{\sigma=(H_0<\cdots<H_m)\in \mathrm{sd}_m(\mathcal{C})} (\mathcal{C}([\phi(H_0)]) \otimes \cdots \otimes \mathcal{C}([\phi(H_m)]))[-m] \\ &\cong \bigoplus_{\sigma\in[G\backslash \mathrm{sd}_m(\mathcal{C})]} (\mathcal{C}([\phi(H_0)]) \otimes \cdots \otimes \mathcal{C}([\phi(H_m)]))[-m] \uparrow^G_{G_\sigma} \\ &\simeq \bigoplus_{\sigma\in[G\backslash \mathrm{sd}_m(\mathcal{C})]} \mathcal{C}(\mathrm{E}(N(H_0)/H_0))[-m] \uparrow^G_{G_\sigma} \end{aligned} \tag{6.21}$$

where we let G act permuting the summands according to the action on $\mathcal{C}$.

Proof. We can identify $\mathcal{C}^m/\mathcal{C}^{m-1}$ as $\oplus_{\sigma\in \mathrm{sd}_m(\mathcal{C})}\mathcal{C}(\sigma)$, where $\mathcal{C}(\sigma)$ is the chain complex of the space of chains spanned by those $\tau \in \mathrm{sd}(P_{\mathcal{C}})$ with $f(\tau) = \sigma$. Such τ biject with the $(m+1)$-tuples $(\rho_0, \cdots, \rho_m)$ where $\rho_i \in \mathrm{sd}[\phi(H_i)]$, where if $\rho_i = \lambda^i_1 \le \lambda^i_2 \le \cdots \le \lambda^i_{n^i}$, then $\tau = (\rho_0 < \rho_1 < \cdots < \rho_m) = \lambda^0_1 \le \lambda^0_2 \le \cdots \le \lambda^0_{n^0} < \lambda^1_1 \le \lambda^1_2 \le \cdots \le \lambda^1_{n^1} < \cdots < \lambda^m_1 \le \lambda^m_2 \le \cdots \le \lambda^m_{n^m}$. The degree in which τ occurs is $m + \sum_{i=1}^m \deg \rho_i$. It follows that $\mathcal{C}(\sigma) \cong (\mathcal{C}([\phi(H_0)]) \otimes \cdots \otimes \mathcal{C}([\phi(H_m)]))[-m]$. Since the map $f\colon P_{\mathcal{C}} \to \mathcal{C}$ is G-equivariant, each $g \in G$ sends $\{\, \tau \in \mathrm{sd}(P_{\mathcal{C}}) \mid f(\tau) = \sigma \,\}$ to $\{\, \tau \in \mathrm{sd}(P_{\mathcal{C}}) \mid f(\tau) = g\sigma \,\}$. Thus G permutes the summands $\mathcal{C}(\sigma)$, which are subcomplexes of $\mathcal{C}^m/\mathcal{C}^{m-1}$. The stabilizer of $\mathcal{C}(\sigma)$ is G_σ, so we have

$$\mathcal{C}^m/\mathcal{C}^{m-1} \cong \bigoplus_{\sigma \in [G \backslash \operatorname{sd}_m(\mathcal{C})]} \mathcal{C}(\sigma)\uparrow_{G_\sigma}^{G} \tag{6.22}$$

Next we have

$$\begin{aligned}(\mathcal{C}([\phi(H_0)]) \otimes \cdots \otimes \mathcal{C}([\phi(H_m)])) &= \mathcal{C}(\mathrm{E}(N(H_0)/H_0)) \otimes \cdots \otimes (\mathrm{E}(N(H_m)/H_m)) \\ &\simeq_{G_\sigma} \mathcal{C}(\mathrm{E}(N(H_0)/H_0) \times \cdots \times \mathrm{E}(N(H_m)/H_m)) \\ &\simeq_{G_\sigma} \mathcal{C}(\mathrm{E}(N(H_0)/H_0)\end{aligned} \tag{6.23}$$

where we have used Proposition 5.6 for the first step. All of the identifications here are evident except perhaps the last one. We claim that

$$\mathrm{E}(N(H_0)/H_0) \times \cdots \times \mathrm{E}(N(H_m)/H_m) \tag{6.24}$$

with the diagonal G_σ-action is equivariantly homotopy equivalent to $\mathrm{E}(N(H_0)/H_0)$, with the action of $N(H_0)$ restricted to G_σ. This is because (6.24) is a contractible space (each factor is contractible) with a free action of G_σ, since the action of G_σ on the first factor is free. $\square$

Let us apply this results to the example of section 6.2. Again, let k be the field of two elements, $G = S_4$, $p = 2$ and $\mathcal{C} = \mathcal{S}_p(G)$. From that section, we know that $\mathcal{C}^0$ is chain homotopy equivalent to the complex in the first column of Figure 6.1. Also, from Proposition 6.5, we get that $(\mathcal{C}^1/\mathcal{C}^0)[-1]$ is precisely the complex in the third column of the mentioned diagram. Hence the sequence of Proposition 6.18 reduces to the short exact sequence in Figure 6.2.

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