The Steinberg Complex of an Arbitrary Finite Group in Arbitrary Positive Characteristic

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Dedication

To my parents.

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Chapter 1

Introduction

The Steinberg complex is a generalization, to the case of an arbitrary finite group, of the Steinberg module of a finite group of Lie type. Given a finite group G, a prime number p, and a field R of characteristic p, one may build the Steinberg complex $\mathrm{St}_p^R(G)$, which is a complex of projective RG-modules. If G happens to be a finite group of Lie type over characteristic p, then the Steinberg complex will be the zero module everywhere, except in a certain dimension, where it will be the Steinberg module. This is why the Steinberg complex deserves its name.

In Chapter 2 of this paper we give some background information and definitions necessary to define the Steinberg complex. Much of this is topological in nature, because the definition of the Steinberg complex involves a simplicial complex Δ with a G-action. Stripping away some extraneous information from Δ then yields the Steinberg complex. This process is completely analogous to the case of a finite group of Lie type, where one defines a topological space called the building, of which the top homology gives the classical Steinberg module.

In Chapter 3 we prove a generalization of an important theorem of Webb. Webb's theorem is the process by which we strip away the extra information from Δ to define the Steinberg complex. It says that if a bounded CW-complex Δ has a G-action with certain properties, then the chain complex of RG-modules $\tilde{C}_*(\Delta; R)$ decomposes as direct sum of two complexes $D_* \oplus P_*$, where D_* is chain-homotopy equivalent to the zero complex and P_* is a complex of projective RG-modules.

We prove that the theorem holds even if Δ is infinite-dimensional, which allows for the possibility of "Steinberg complex analogues" coming from a new class of CW-complexes. This proof introduces functors, called coefficient systems, which are defined on the category of G-sets and which themselves form an abelian category. At the end of the argument, one evaluates an entire complex of these functors, yielding the Steinberg complex of RG-modules. The proof uses category theory and representation theory of categories, as well as a great deal of homological algebra.

In Chapter 4 we consider the case of some infinite-dimensional CW-complexes which have been defined and which are known to have the same homology as the Steinberg complex. We show that the "Steinberg complex analogues" coming from these new CW-complexes are in fact chain homotopy equivalent to the Steinberg complex, and that this will always happen. This is proven using homological algebra.

In Chapters 5 through 9 we focus on calculating a concrete example of a Steinberg complex of a group from beginning to end, to see what properties the Steinberg complex might have. We are interested in understanding a case in which the Steinberg complex, which is a complex of projective RG-modules, has homology in some degrees which is nonprojective. This seems to be a rare phenomenon, and the simplest case we can find to investigate is a group of order $3^4 * 7^3 = 27,783$. It takes a combination of different methods – group theory, representation theory, topology, and computer programming – to analyze this example fully.

As it turns out, this example also proves that the Steinberg complex need not be what is called a (partial) tilting complex.

In Chapters 10 and 11, we investigate another example of a Steinberg complex with non-projective homology, although we do not analyze this one as thoroughly. Instead, we prove that it is the smallest group of order $p^a q^b$ which holds this property. Since this group has order $2^5 * 3^4 = 2,592$, there are a lot of smaller groups to eliminate. This proof makes substantial use of results from group theory, especially a pair of theorems of Burnside and Glauberman, but also relies on topology and representation theory, as well as brute-force computing to eliminate some difficult cases.

Finally, in Chapter 12 we examine the category of coefficient systems, which arise in Chapter 3 with the proof of Webb's Theorem and its generalization. After a certain amount of background development, we are able to prove that the complex $R[\Delta^?]$ of functors appearing in Chapter 3 does satisfy a "tilting complex" property that the

Steinberg complex (which is obtained by uniformly evaluating the complex $R[\Delta^?])$ lacks.

Chapter 2

Subgroup complexes

Definition 2.1 *Let G be a group. A G-poset is a partially-ordered set Y together with a G-action, such that for all $a, b \in Y$ and for all $g \in G$, if $a \leq b$ then $g \cdot a \leq g \cdot b$.*

A map of G-sets is a function $f : X \to Y$ where X and Y are G-sets, such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. A map of posets is a function $f : X \to Y$ where X and Y are posets, such that for all $x, y \in X$, if $x \leq y$ then $f(x) \leq f(y)$. A map of G-posets is a function $X \to Y$ where X and Y are G-posets, such that f is a map of G-sets and a map of posets.

Example: Let G be a finite group, and p a prime. We will write $\mathcal{S}_p = \mathcal{S}_p(G)$ to denote the poset of nontrivial p-subgroups of G, with the order relation given by inclusion of subgroups. Then $\mathcal{S}_p$ is a G-poset under the (left) G-action given by conjugation.

Definition 2.2 *Let G be a group. A G-simplicial complex is a simplicial complex Δ together with a G-action, such that for each simplex σ of Δ and each $g \in G$, the image $g \cdot \sigma$ is a simplex of G, and if g fixes σ setwise then g fixes σ pointwise.*

A map of simplicial complexes is a function $f : X \to Y$ where X and Y are simplicial complexes, such that f sends each simplex of X to a simplex of Y by a linear map which maps vertices to vertices. A map of G-simplicial complexes is a function $f : X \to Y$ where X and Y are G-simplicial complexes, such that f is a map of simplicial complexes and f is a map of G-sets.

Now let G be a finite group, and Y a G-poset. We define the *order complex* of Y to be the simplicial complex whose n-simplices are exactly the totally-ordered subsets

$$P_0 < P_1 < \ldots < P_n$$

in Y, for each $n \geq 0$. Then Y is a G-simplicial complex.

Further, a map $f : X \to Y$ of G-posets induces a map $|f| : |X| \to |Y|$ of G-simplicial complexes.

Definition 2.3 *Let X and Y be G-simplicial complexes, and let $\alpha, \beta : X \to Y$ be maps of G-simplicial complexes. We say that α and beta are G-homotopic if there is a map $F : X \times I \to Y$ of G-simplicial complexes, where I is the closed interval $[0, 1]$ with trivial G-action, such that $F(x, 0) = \alpha(x)$ and $F(x, 1) = \beta(x)$ for all $x \in X$. In this case F is called a G-homotopy.*

Example: If G is a non-identity p-group, then $|\mathcal{S}_p|$ is contractible, by a homotopy which drags every vertex $P_0 < G$ along its edge to the vertex G, and leaves the vertex G fixed throughout. It turns out that we may choose this homotopy to be a G-homotopy. Let $X = |\mathcal{S}_p|$, and let $\alpha : X \to \{*\}$ be the unique map, and let $\beta : \{*\} \to X$ be the map which sends the point $*$ to the vertex G. Of course $\alpha \circ \beta$ is the identity on $\{*\}$, so we just need to find a homotopy between the constant map $\beta \circ \alpha : x \mapsto G$ and the identity map $x \mapsto x$. We let $F(x, t) = (1 - t) * x + t * G$, using barycentric coordinates. Then checking the definitions shows that F is a G-homotopy.

This example may be generalized. Recall that $O_p(G)$ denotes the maximal normal p-subgroup of G, which is given by the intersection of all Sylow p-subgroups of G. Then:

Theorem 2.4 *([33] 2.4). If $O_p(G) \neq 1$, then $|\mathcal{S}_p|$ is G-contractible.*

Proof: A G-contraction takes a vertex P and moves it first up to the vertex $PO_p(G)$, then down to $O_p(G)$. Note that $PO_p(G)$ is a nonidentity p-subgroup because $O_p(G)$ is normal (and P and $O_p(G)$ are nonidentity p-subgroups). $\square$

The converse is also true: If $\mathcal{S}_p$ is G-contractible, then $O_p(G) \neq 1$. Quillen conjectured a stronger statement: If $\mathcal{S}_p$ is contractible (not necessarily G-contractible), then $O_p(G) \neq 1$. Quillen's Conjecture has been proven in many cases but remains unproven in general. Note that for the proof above we needed $O_p(G)$ to be non-trivial so that it would be a vertex in $|\mathcal{S}_p|$.

In proving this theorem, Quillen introduced another complex, $\mathcal{A}_p(G)$, which we wish to define now along with several other complexes.

Definition 2.5 *(i) Let $\mathcal{A}_p(G)$ denote the poset of nontrivial elementary abelian p-subgroups of G. This is the subposet of $\mathcal{S}_p(G)$ whose objects are the subgroups of G isomorphic to a direct product of a positive number of cyclic groups of order p.*

(ii) Let $\mathcal{Z}_p(G)$ denote the subposet of $\mathcal{A}_p(G)$ consisting of objects P satisfying $P = \langle\{x \in O_p(Z(C_G(P))) : x^p = 1\}\rangle$.

(iii) Let $\mathcal{B}_p(G)$ denote the subposet of $\mathcal{S}_p(G)$ consisting of objects P satisfying $P = O_p(N_G(P))$.

(iv) Let $\mathcal{R}_p(G)$ denote the simplicial complex whose vertices are the nonidentity p-subgroups of G and whose n-simplices are exactly the chains of normal *inclusions, meaning that the n-simplices are all chains of the form $K_0 < K_1 < \ldots < K_n$ such that $K_i \lhd K_n$ for all i. (Note that the normal subgroup relation is not transitive. However, $A \lhd C$ and $A < B < C$ imply $A \lhd B$, so that each nonempty subset of the vertices of a simplex is itself a simplex.)*

(v) [1] Let $\mathcal{C}_p(G)$ denote the complex whose vertices are the subgroups of G of order p and whose n-simplices are the sets of $n+1$ such subgroups which centralize each other.

Theorem 2.6 *([1], [9], [33], [6], [49]). $|\mathcal{S}_p(G)|$, $|\mathcal{A}_p(G)|$, $|\mathcal{Z}_p(G)|$, $|\mathcal{B}_p(G)|$, $\mathcal{R}_p(G)$, and $\mathcal{C}_p(G)$ are all G-homotopy-equivalent.*

This theorem allows us to consider whichever of these G-equivalent spaces is most convenient for the purpose at hand: often using the smaller posets $\mathcal{A}_p$, $\mathcal{B}_p$, and $\mathcal{R}_p$ can make calculation substantially easier. Further, theorems like the one of Quillen's above may be proven just once rather than several times.

As an example, take $G = D_{12}$ to be the dihedral group with 12 elements, and $p = 2$. D_{12} has 3 Sylow 2-subgroups isomorphic to V_4 (one of these is generated by a horizontal and a vertical flip of the hexagon). Each copy of V_4 has 3 cyclic subgroups of order 2, and these are all the 2-subgroups of D_{12}. Moreover, the Sylow subgroups all intersect in a cyclic group Y of order 2, generated by the 180-degree rotation. In each of the three simplicial complexes in Figure 2.1, the distinguished vertex is this Y. Note: the three "claws" in the third complex are meant to be the same, even though one is drawn above the other two.

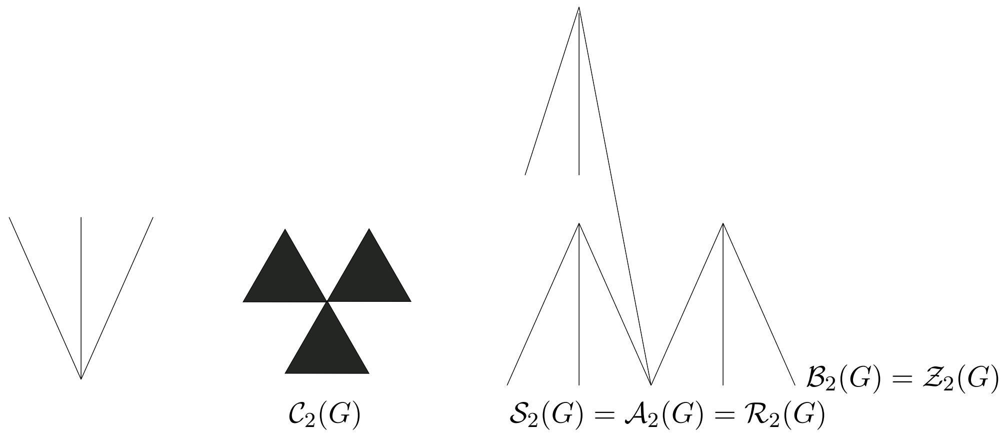

Figure 2.1: Subgroup complexes for $G = D_{12}$ at $p = 2$

It is not too surprising that $\mathcal{S}_p, \mathcal{A}_p$, and even $\mathcal{R}_p$ have been defined, but $\mathcal{B}_p$ needs some explanation. Its motivation comes from considering the case where G is a Chevalley group in characteristic p: in this case the condition $P = O_p(N(P))$ is satisfied exactly when P is the unipotent radical of a parabolic subgroup. But the operation of taking the unipotent radical of a subgroup is inclusion-reversing. Hence $\mathcal{B}_p$ is the opposite poset of the poset of parabolic subgroups; *i.e.*, it is the opposite of the building. But a poset and its opposite yield equivalent simplicial complexes.

Thus $\mathcal{B}_p(G)$ generalizes to an arbitrary finite group the building of a finite group of Lie type. The Solomon-Tits Theorem states that the building of a finite group of Lie type is G-homotopy-equivalent to a bouquet of spheres of dimension one less than the semisimple rank of G. The top homology of the building may be defined to be the Steinberg module of G. We have:

Theorem 2.7 *If G is a finite group of Lie type of characteristic p (or more generally, is a finite group with a split BN-pair of rank ≥ 2 in characteristic p), then $\mathcal{B}_p(G)$ is homotopy-equivalent to the Steinberg module concentrated in degree one less than the semisimple rank of G.*

So $\mathcal{B}_p(G)$, and the other subgroup complexes, generalize the notion of the Steinberg module of a finite group of Lie type, although for an arbitrary finite group, of course,

$\mathcal{B}_p(G)$ may be a complex with reduced homology in more than one dimension. Some people therefore call these complexes "Steinberg complexes," but strictly speaking we shall save that name for a closely-related complex, defined in the next section.

We wish to point out one more fact about these subgroup complexes, which we shall need later on.

Proposition 2.8 *([33]).* $|\mathcal{S}_p(G \times H)| \simeq |G| \star |H|$, *where* $|G| \star |H|$ *represents the topological join of the spaces* $|G|$ *and* $|H|$.

Finding geometric objects (in this case, complexes) on which a group G acts is a proven approach to learning about G. The original motivation in studying these p-subgroup complexes seems to have been in the context of group cohomology (see for instance [33], [52]). Computational results suggested that certain subgroup complexes like $\mathcal{A}_p$ and $\mathcal{B}_p$ would produce good formulas for calculating cohomology groups.

In the next section, given a G-simplicial complex $|Y|$, we will study the augmented chain complex $\tilde{C}_*(|Y|, k)$, where k is a commutative ring with identity.

Example: If p does not divide the order of G, then $\mathcal{S}_p$ is empty (since we exclude the trivial subgroup). Therefore $\tilde{C}_*(|\mathcal{S}_p|; \mathbb{Z})$ is zero everywhere except in degree -1, where it is the trivial module $\mathbb{Z}$.

To close this section, we point out that the conjugation action of G on $\mathcal{S}_p(G)$ induces the structure of a complex of kG-modules onto $\tilde{X}_* = \tilde{C}_*(|\mathcal{S}_p|; k)$. In fact, each $\tilde{X}_n$ is a kG-permutation module.

To see this, we simply need to observe that the vertices of $|\mathcal{S}_p|$ are p-subgroups of G, and G acts on the set of p-subgroups of G by conjugation. The higher-dimensional simplices are longer chains of inclusions of p-subgroups, and since conjugation respects inclusion, it follows that G acts on the set of n-simplices of $|\mathcal{S}_p|$ for each $n \geq 0$. Also, $\tilde{X}_{-1}$ is a single copy of k, which is a free k-module on one generator, and there is exactly one G-action on a one-element set. Thus for each n, G permutes the basis elements of $\tilde{X}_n$.

For each element $g \in G$, we extend this action to be k-linear, and we check that $\tilde{X}_n$ is a kG-module for each n. Finally we check that each boundary homomorphism $d_n : \tilde{X}_n \to \tilde{X}_{n-1}$ commutes with conjugation by each $g \in G$, and from this it follows that d_n is a map of kG-modules. $\square$

This means that the homology in each dimension will also be a kG-module, though not necessarily a kG-permutation module.

We shall prove in the next section that, for instance, the chain complex $\tilde{C}_*(\mathcal{S}_p;\mathbb{F}_p)$ has a very interesting form, which will have theoretical and computational applications.

Chapter 3

The Steinberg complex: existence

This section is devoted to proving quite a deep and surprising theorem. Webb, followed by others, proved it in the case of bounded complexes ([53], [47], [9]). We have here extended it to include the case where the complex need not be bounded – that is, it may have simplices in infinitely many dimensions. We shall need the infinite-dimensional case of this theorem later on, when we touch on the complexes described by Dwyer [16].

Theorem 3.1 *(Main Structure Theorem.) Let G be a finite group, p a prime, and R a complete p-local ring. Further, let Σ be any (possibly infinite-dimensional) G-CW-complex having only finitely many cells in each degree, such that for each nonidentity p-subgroup $Q \leq G$, the subcomplex Σ^Q consisting of cells fixed by every element of Q is contractible. Then:*

(i) $\tilde{C}_*(\Sigma; R) \cong D_* \oplus J_*$, where D_* is RG-chain homotopy equivalent to the zero complex and J_* is a complex of projective RG-modules. This is an isomorphism of complexes of RG-modules, so in particular $\tilde{C}_*(\Sigma; R)$ is RG-chain homotopy equivalent to J_*.

(ii) Up to isomorphism of complexes, there is a unique minimal summand J_* satisfying part (i).

A complete p-local ring is either a field of characteristic p or a complete discrete valuation ring with residue field of characteristic p. In particular, the p-element field $\mathbb{F}_p$ and the ring $\mathbb{Z}_p$ of p-adic integers are complete p-local rings.

Note: if k is of characteristic p and p does not divide $|G|$, then kG is semisimple, and every simple module is projective. In this case part (i) of the Main Structure Theorem is trivial, and (ii) also will be true by the argument below. Thus we shall assume for the rest of this section that p divides $|G|$.

Proposition 3.2 *If Σ is the G-simplicial complex $|\mathcal{S}_p(G)|$, then Σ satisfies the hypotheses of the Main Structure Theorem.*

Remark: Essentially, a G-CW-complex is CW-complex with a G-action, so that for each $g \in G$, the action of g is a cellular map. A G-simplicial complex is an example of a G-CW-complex. The complexes we study in this paper are all simplicial complexes, but there is no reason not to state the theorem in a more general setting.

In view of Proposition 3.2, Theorem 3.1 says that if we take our ring R of coefficients to be, say, $\mathbb{F}_p$ or $\mathbb{Z}_p$, then the chain complex of RG-modules $\tilde{C}_*(|\mathcal{S}_p(G)|; R)$ has a direct sum decomposition into a contractible complex and a complex of projectives. (Note: our chain complexes will generally be reduced, and so by "contractible" we will generally mean "chain homotopic to the zero complex".)

Proof of Proposition 3.2: Let G be a finite group, p a prime, and Q a nonidentity p-subgroup of G, and let $\Sigma = |\mathcal{S}_p(G)|$. Then Q is a vertex of Σ^Q: first, Q is a vertex of Σ, and second, if $q \in Q$ then ${}^qQ = Q$. Now suppose P is a vertex of Σ^Q, so P is fixed by Q under conjugation, or in other words $Q \leq \mathrm{Stab}_G(P) = N_G(P)$. Then, as in Quillen's theorem, a contraction of Σ is given by moving each vertex P along an edge to PQ, and then along an edge to Q. $\square$

Remark: Webb's original theorem for a bounded G-CW-complex applies to the case of $|\mathcal{S}_p(G)|$ for a finite group G, because the poset $|\mathcal{S}_p(G)|$ is finite.

Definition 3.3 *When $\Sigma = |\mathcal{S}_p(G)|$, we call such a minimal J_* as in part (ii) of the Main Structure Theorem the* Steinberg complex *of G at p over R, and denote it by $St^R(G)$. If $R = \mathbb{F}_p$, we may write $St^p(G) = St^R(G)$.*

Webb's proof of the original theorem uses properties of Mackey functors, which we do not address here. We give a different proof instead, drawing upon a proof by Peter Symonds of the finite-dimensional case. Serge Bouc ([9]) has also given a proof of that case.

As an application of the Main Structure Theorem, we prove a theorem of Kenneth Brown:

Theorem 3.4 *([7] section 3, Cor. 2) The Euler characteristic $\chi(|\mathcal{S}_p(G)|)$ is congruent to 1, modulo the size of a Sylow p-subgroup of G.*

In the special case in which G is divisible by p but not by p^2, this is one of Sylow's Theorems: in this case $|\mathcal{S}_p(G)|$ will have only 0-simplices, so its Euler characteristic will be equal to the number of subgroups of order p, which are the Sylow p-subgroups.

Brown states this theorem as a corollary of a more general statement about arbitrary (possibly infinite) groups. We will prove it instead as a consequence of Theorem 3.1.

Proof of Theorem 3.4: To agree with Brown, we use the familiar convention that the Euler characteristic of a point is 1, not 0. In other words, for this purpose we agree that our chain complexes will be unreduced. This has the effect that the contractible complex D_* in the Main Structure Theorem will have Euler characteristic 1, not 0.

$$\begin{aligned}
\chi(|\mathcal{S}_p|) &= \Sigma_{n=0}^{d}(-1)^n(\mathrm{rank}_{\mathbb{Z}}(\mathcal{S}_p)_n) \\
&= \Sigma_{n=0}^{d}(-1)^n(\dim_{\mathbb{F}_p}(P_n \oplus D_n)) \\
&= \Sigma_{n=0}^{d}(-1)^n(\dim_{\mathbb{F}_p}(P_n)) + (-1)^n(\dim_{\mathbb{F}_p}(D_n)) \\
&= \chi(P_*) + \chi(D_*) \\
&= \chi(P_*) + 1,
\end{aligned}$$

so it remains to show that $\chi(P_*)$ is divisible by the order of a p-Sylow subgroup.

But the result then follows immediately since each projective kG-module for k a field of characteristic p will have k-dimension divisible by the order of a Sylow p-subgroup of G (see for example [55], Corollary 8.3), and the Euler characteristic is calculated by taking sums and differences of these numbers.

We devote the rest of this section to a proof of Theorem 3.1. We start by following a strategy laid out by Symonds (see [47], sections 2 and 6, and also [45]).

Assume p is a prime, G a finite group, and R a complete p-local ring. We assume p divides $|G|$, because otherwise the theorem is trivial. We need the following definition:

Definition 3.5 *Given a nonempty set $\mathcal{W}$ of subgroups of G which is closed under* *conjugation, we define a* coefficient system *of G over R to be a contravariant functor F from the category of finite G-sets with stabilizers in $\mathcal{W}$ to the category of R-modules, which respects finite direct sums, meaning that F sends the disjoint union $\bigoplus_{i=1}^{k} G/H_i = \amalg_{i=1}^{k} G/H_i$ to the direct sum $\bigoplus_{i=1}^{k} F(G/H_i)$. We define $CS_{\mathcal{W}}(G)$ to be the category with objects the coefficient systems of G over R and $Hom_{CS_{\mathcal{W}(G)}}(F,G) = \{natural\ transformations : F \to G\}$.*

In particular, suppose G is a finite group and $\mathcal{W}$ is as above. If $H \in \mathcal{W}$, then G/H is a finite G-set whose stabilizers all lie in $\mathcal{W}$ – more precisely, $\mathrm{Stab}_G(tH) = {}^tH$ for all $t \in G$. So if we fix a G, a $\mathcal{W}$, and an $H \in \mathcal{W}$, then every $F \in CS_{\mathcal{W}}(G)$ gives an R-module $F(G/H)$, which we could call $F(H)$ without fear of confusion.

Also, whenever $K \leq H$ and $K, H \in \mathcal{W}$, we get a G-set map $G/K \to G/H$ and therefore F determines a map $F(H) = F(G/H) \to F(G/K) = F(K)$. And finally for each $g \in G, H \in \mathcal{W}$ we have a G-set map $c_g : G/H \to G/H^{g^{-1}}$, given by

$$c_g(tH) = tHg^{-1} = tg^{-1}gHg^{-1} = tg^{-1}H^{g^{-1}}.$$

Therefore F determines a map $F(H^{g^{-1}}) = F(G/H^{g^{-1}}) \to F(G/H) = F(H)$.

Conversely, given $F(H), F(H \leq K), F(c_g)$ for all $K \leq H$ and for all c_g, we can recover the coefficient system. We send an arbitrary finite G-set, isomorphic to $G/H_1 \oplus \ldots \oplus G/H_n$ for some n, to $F(H_1) \oplus \ldots \oplus F(H_n)$. And any homomorphism $G/H \to G/K$ of transitive G-sets may be factored as $G/H \to G/J \to G/K$, where $H \leq J$, $K = g^{-1}Jg$ for some $g \in G$, and the maps are given by $xH \mapsto xJ$ and $xJ \mapsto xJg = xg(g^{-1}Jg)$.

This means that $CS_{\mathcal{W}}$ is equivalent to the category of contravariant functors $\mathcal{W} \to R$-mod, where the morphisms of $\mathcal{W}$ are generated by the inclusion and conjugation maps. There is a subtlety here: the conjugation maps are in bijection with the conjugation maps $G/H \to G/H^g$ in the category of G-sets, not the maps $H \to H^g$. This is an important distinction – for example, if $H = 1$ is the identity subgroup of G, then there is only one map $1 \to 1^g$, namely the identity map, since 1 is normal in G. However, the G-set maps $G/1 \to G/1^g$ are those which send $x1 \mapsto x1g = xg1$, so that these maps form a group isomorphic to G (or G acting regularly on itself).

It is extremely convenient to observe that $CS_{\mathcal{W}}(G)$ is in fact equivalent to the category mod-$R\mathcal{W}$ of all right $R\mathcal{W}$-modules, where $R\mathcal{W}$ is the category algebra of $\mathcal{W}$ ([54], Propositions 2.1 and 2.2). This identification uses the fact that $\mathcal{W}$ is a finite category – otherwise we might get only a full embedding into mod-$R\mathcal{W}$. (We get right modules because we have defined $CS_{\mathcal{W}}(G)$ to have contravariant functors, but this is not a crucial point.)

For a (finite) G-set X, we define $R[X^?]$ to be the coefficient system determined by associating to each $H \in \mathcal{W}$ the free R-module $R[X^H]$ with generators the fixed points of X under H. Here $H \leq J$ is sent to $R[X^J] \subseteq R[X^H]$, and $c_g : J \to g^{-1}Jg$ is sent to the R-module map $R[X^{g^{-1}Jg}] \to R[X^J]$ given on generators by $\sigma \mapsto g \cdot \sigma$.) All of the coefficient systems we use here are of the form $R[X^?]$ for some X. Often we shall have $X = G/H$ for a fixed subgroup $H \leq G$.

We remark here that $R[(G/H)^K]$ is naturally an $R[N_G(H)]$-module and not just an R-module, by the left action

$$n \cdot gH = gHn^{-1} = gn^{-1}H,$$

for all $gH \in (G/H)^K$ and $n \in N_G(H)$. Further, under this action H acts trivially on $R[(G/H)^K]$, so $R[(G/H)^K]$ is an $R[N_G(H)/H]$-module. In fact for any $L \in CS_{\mathcal{W}}(G)$, $L(H)$ is an $R[N_G(H)/H]$-module: if $x \in L(H)$ and $n \in N_G(H)$, then define $nH \cdot x := L(c_n)(x)$, where $L(c_n) : L(H) \to L(H^{n^{-1}}) = L(H)$ is the map induced by the conjugation map $c_n : H \to H^{n^{-1}}$.

Now, given $\emptyset \neq \mathcal{V} \subseteq \mathcal{W}$ we define $\mathrm{Res}^{\mathcal{W}}_{\mathcal{V}} : CS_{\mathcal{W}}(G) \to CS_{\mathcal{V}}(G)$ to be the forgetful functor $F \mapsto F \circ i$, where $i : \{G\text{-sets with stabilizers in } \mathcal{V}\} \to \{G\text{-sets with stabilizers in } \mathcal{W}\}$ is inclusion.

Proposition 3.6 *For each $H \in \mathcal{W}$ and for each coefficient system $L \in CS_{\mathcal{W}}$, the map $\alpha(\eta) = \eta_H(eH)$ is an R-module isomorphism*

$$\alpha : Hom_{CS_{\mathcal{W}}}(R[(G/H)^?], L) \xrightarrow{\sim} L(H).$$

Remark: This is essentially the contravariant form of Yoneda's Lemma, with the addition that everything takes place in the category of R-modules rather than the category of sets. Rather than appealing to the usual version of Yoneda's Lemma, however, we will just prove our result directly.

Proof: Certainly α is a map of R-modules, so we need only show that it is a bijection. Let $K \in \mathcal{W}$ and write $S = R[(G/H)^?]$.

To show that α is surjective, we let $u \in L(H)$. Then we need to define a natural transformation $\eta : R[(G/H)^?] \to L$. We know that K fixes gH if and only if $g^{-1}Kg \leq H$. Thus if K is not G-conjugate to a subgroup of K, then $R[(G/H)^K] = 0$ and in this case η_K must be 0.

On the other hand, suppose there exists a $g \in G$ such that $K^g \leq H$. Then we have a pair of G-set maps

$$G/H \xleftarrow{\pi} G/K^g \xleftarrow{c_g} G/K,$$

which give rise to R-module maps

$$\begin{array}{ccccc} R[(G/H)^H] & \longrightarrow & R[(G/H)^{K^g}] & \longrightarrow & R[(G/H)^K], \\ \\ tH & \mapsto & tH & \mapsto & gtH \end{array}$$

Thus if $gH \in R[(G/H)^K]$, and if η is a natural transformation, then we must have $\eta_K(gH) = \eta_K \circ S(\pi \circ c_g)(eH) = L(\pi \circ c_g) \circ \eta_H(eH)$. This shows that η is completely determined by the value of $\eta_H(eH)$, and therefore α is injective.

To show that α is surjective, we must show that for every $u \in L(H)$, the choice $\eta_H(eH) = u$ determines a well-defined natural transformation η. First, we show that η is well-defined. Suppose that $gH = tH \in (G/H)^K$. Then we have two chains of G-set maps:

$$\begin{array}{ccccc} G/K & \xrightarrow{\pi_H^{K^g}} & G/K^g & \xrightarrow{c_g} & G/H, \\ \\ sK & \mapsto & sgK^g & \mapsto & sgH, \end{array}$$

and

$$\begin{array}{ccccc} G/K & \xrightarrow{\pi_H^{K^t}} & G/K^t & \xrightarrow{c_t} & G/H, \\ \\ sK & \mapsto & stK^t & \mapsto & stH. \end{array}$$

Now, $\eta_K(gH) = L(\pi_H^{K^g} \circ c_g)(\eta_H(eH)) = L(\pi_H^{K^g} \circ c_g)(u)$, and similarly $\eta_K(tH) = L(\pi_H^{K^t} \circ c_t)(u)$. However, we see that $\pi_H^{K^g} \circ c_g : sK \to sgH$ for all $sK \in G/K$, and $\pi_H^{K^t} \circ c_t :$

$sK \to stH$ for all $sK \in G/K$. Thus, since $gH = tH$ by assumption, we have that

$$\pi_H^{K^g} \circ c_g = \pi_H^{K^t} \circ c_t : G/K \to G/H,$$

and therefore

$$L(\pi_H^{K^g} \circ c_g)(u) = L(\pi_H^{K^t} \circ c_t)(u).$$

This proves that η is well-defined.

Finally we need to show that η is a natural transformation (*i.e.*, that naturality holds even for commutative squares that do not involve $R[(G/H)^H]$). Suppose that $f : G/J \to G/K$ is a G-set map. We need the diagram in Figure 3.1 to commute:

$$\begin{tikzcd}
R[(G/H)^K] \arrow[r, "S(f)"] \arrow[d, "\eta_K"] & R[(G/H)^J] \arrow[d, "\eta_J"] \\
L(K) \arrow[r, "L(f)"] & L(J)
\end{tikzcd}$$

Figure 3.1: The natural transformation condition

If $R[(G/H)^K] = 0$ then the diagram commutes, so we assume that there exists some $gH \in (G/H)^K$. Then $K^g \leq H$, and there is a G-set map $p : G/K \to G/H$. Also, since $f : G/J \to G/K$ is a G-set map, $J^t \leq K$ for some $t \in G$, and $J^{tg} \leq H$. As before,

$$\eta_K(gH) = L(p)(u).$$

Now

$$\begin{aligned}
(\eta_J \circ S(f))(gH) &= \eta_J(tgH) \\
&= \eta_J \circ S(f) \circ S(p)(eH) \\
&= \eta_J \circ S(p \circ f)(eH) \\
&= L(p \circ f) \circ \eta_H(eH) \\
&= L(p \circ f)(u) \\
&= L(f) \circ L(p)(u) \\
&= L(f) \circ \eta_K(gH). \square
\end{aligned}$$

Proposition 3.7 *$R[(G/H)^?]$ is projective in $CS_{\mathcal{W}}(G)$ if $H \in \mathcal{W}$.*

Proof: Let $\theta : M \to N$ be an epimorphism and $\eta : R[(G/H)^?] \to N$ be a morphism in $CS_{\mathcal{W}}(G)$. Regard $CS_{\mathcal{W}}(G)$ as mod-$R\mathcal{W}$. Then by the proof of Proposition 2.1 in [54], each functor L becomes $\bigoplus_{K\in\mathcal{W}} L(K)$ and every natural transformation α becomes $\bigoplus_{K\in\mathcal{W}} \alpha_K$. In particular, the epimorphism θ is a surjective $R\mathcal{W}$-map equal to $\bigoplus_{K\in\mathcal{W}} \theta_K$.

Since θ is a surjection there exists $u \in M(H)$ such that $\theta(u) = \eta_H(eH) \in N(H)$, and since θ_K maps into $N(K)$ for each K, it must be that $u \in M(H)$ and $\theta_H(u) = \eta_H(eH)$.

Now let $\phi : \mathrm{Hom}_{CS_{\mathcal{W}}}(R[(G/H)^?], M) \to M(H)$ denote the R-module isomorphism $\phi(\zeta) = \zeta_H(eH)$ for all η, guaranteed by the last proposition. Thus

$$\phi^{-1}(u) \in \mathrm{Hom}_{CS_{\mathcal{W}}}(R[(G/H)^?], M),$$

and

$$\begin{aligned} u &= \phi(\phi^{-1}(u)) \\ &= (\phi^{-1}(u))_H(eH). \end{aligned}$$

Thus we get

$$\begin{aligned} \eta_H(eH) &= \theta_H(u) \\ &= \theta_H(((\phi^{-1}(u))_H)(eH)) \\ &= (\theta_H \circ (\phi^{-1}(u))_H))(eH) \\ &= (\theta \circ (\phi^{-1}(u))_H(eH), \end{aligned}$$

which implies as in Proposition 3.6 that $\eta = \theta \circ \phi^{-1}(u)$. Thus $R[(G/H)^?]$ is projective. $\square$

We now define, for a G-CW complex Ω and a subposet $\mathcal{X} \subseteq \mathcal{S} = \mathcal{S}(G)$ such that $\mathcal{X}$ is closed under conjugation, a chain complex $C_*(\Omega^?)$ of objects in $CS_{\mathcal{X}}(G)$. We define $C_n(\Omega^?) = R[\Omega_n^?]$, where Ω_n is the G-set of n-cells, and the boundary map $d_n : C_n(\Omega^?) \to C_{n-1}(\Omega^?)$ by the "obvious" natural transformation – for each $H \in \mathcal{X}$ and for all $\omega \in \Omega^H$, we set

$$(d_n)_H(\omega) = \delta_n(\omega),$$

where δ_n is the n^{th} boundary homomorphism in the usual chain complex $C_*(\Omega; R)$. Note that $\tilde{C}_n(\Omega^?)$ is a complex of projective objects of $CS_{\mathcal{S}}(G)$, since $\mathcal{S} = \mathcal{S}(G)$ is the set of all subgroups of G and therefore $\mathrm{Stab}_G(\omega) \in \mathcal{S}$ for all $\omega \in \Omega$.

Now let F be a Sylow p-subgroup of G. By the Note following the statement of the Main Structure Theorem, $F \neq 1$. Also, let $\Delta = \Sigma$ be the G-CW-complex in the statement of Theorem 3.1, let $\mathcal{S} = \mathcal{S}(F)$ be the set of all subgroups of F, let $\mathcal{V} = \mathcal{S} - \{1\}$ be the set of all nonidentity subgroups of $F \neq 1$, and let Δ_S be the subcomplex of Δ consisting of cells $\delta \in \Delta$ such that $\mathrm{Stab}_F(\delta) \in \mathcal{V}$.

First, we note that F acts on Δ_S: if $1 \neq f \in \mathrm{Stab}_F(\delta)$ and $h \in F$ then $1 \neq hfh^{-1} \in F$ and $hfh^{-1}(h\delta) = hf\delta = h\delta$, so $\mathrm{Stab}_F(h\delta) \neq 1$ for all $\delta \in \Delta_S, h \in F$. Thus Δ_S is an F-subcomplex of Δ.

We get the usual short exact sequence of chain complexes:

$$0 \to \tilde{C}_*(\Delta_S^?) \to \tilde{C}_*(\Delta^?) \to C_*((\Delta, \Delta_S)^?) \to 0.$$

Apply $\mathrm{Res}_{\mathcal{S}}^{\mathcal{V}}$ to $\tilde{C}_*(\Delta_S^?)$ to get a chain complex in $CS_{\mathcal{V}}(F)$. This complex is acyclic if and only if for every $X \in \mathcal{V}$, the complex $\tilde{C}_*(\Delta_S^X)$ obtained by evaluating at X is acyclic. Let $X \in \mathcal{V} = \mathcal{S}(F) - \{1\}$, or equivalently $1 \neq X \leq F$.

Then

$$\begin{aligned}
\Delta_S^X &= \{\sigma \in \Delta_S | x \cdot \sigma = \sigma, \forall x \in X\} \\
&= \{\sigma \in \Delta | \mathrm{Stab}_F(\sigma) \neq 1, x \cdot \sigma = \sigma, \forall x \in X\} \\
&= \{\sigma \in \Delta | 1 \neq \mathrm{Stab}_F(\sigma) \geq X\} \\
&= \{\sigma \in \Delta | \mathrm{Stab}_F(\sigma) \geq X \neq 1\} \\
&= \{\sigma \in \Delta | \mathrm{Stab}_F(\sigma) \geq X\} \\
&= \Delta^X.
\end{aligned}$$

X is a nontrivial p-subgroup of G, and so to verify that $\tilde{C}_*(\Delta_S^?)$ is acyclic we just need to know that Δ^P is R-acyclic for each nontrivial p-subgroup P of G, which is true by the hypotheses on Σ in the statement of Theorem 3.1.

In addition $\tilde{C}_*(\Delta_S^?)$ consists of projective objects of $CS_{\mathcal{V}}(F)$, because as F-sets $\Delta_{S,n} \cong \bigoplus_{i=1}^k (F/H_i)$, and therefore as $R\mathcal{V}$-modules $R[(\Delta_{S,n})^?] \cong \bigoplus_{i=1}^k R[(F/H_i)^?]$ is a direct sum of projective modules since each $\delta \in \Delta_{S,n}$ has stabilizers in $\mathcal{V}$. Thus if we consider it as a complex of right $R\mathcal{V}$-modules, then $\tilde{C}_*(\Delta_S^?)$ is an acyclic complex of projective modules, and therefore is a contractible complex.

This means that there are $s_n : \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}\tilde{C}_n \to \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}\tilde{C}_{n+1}$ such that $(\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}d_{n+1})s_n + s_{n-1}\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}d_n = \mathrm{Id}_{\mathrm{Res}\tilde{C}_n} = \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}\mathrm{Id}_{\tilde{C}_n}$. We wish to show that this implies $\tilde{C}_*(\Delta_S^?)$ is a contractible complex of $R\mathcal{S}$-modules. Note that $R\mathcal{V}$ is a subalgebra of $R\mathcal{S}$, so in general $R\mathcal{V}$-maps may not be $R\mathcal{S}$-maps.

Since $\tilde{C}_n(\Delta_S^?) \cong \bigoplus R[(F/H_i)^?]$ with $H_i \in \mathcal{V}$, it suffices to assume $\tilde{C}_n(\Delta_S^?) = R[(F/H)^?]$ for some $H \in \mathcal{V}$. Now for each $H \in \mathcal{V} \subseteq \mathcal{S}$, we have:

$$\mathrm{Hom}_{\mathcal{V}}(\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}R[(F/H)^?], \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}T) = \mathrm{Hom}_{\mathcal{V}}(R[(F/H)^?], \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}T)$$

$$\cong \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}T(H) = T(H) \cong \mathrm{Hom}_{\mathcal{S}}(R[(F/H)^?], T)$$

and this isomorphism sends $f \mapsto \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}f$ since the left-hand isomorphism sends $(\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}f)$ to $(\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}f)_H(eH) = f_H(eH)$, which is the image under the right-hand isomorphism of f. Thus for all $H \in \mathcal{V}$ and $T \in CS_{\mathcal{S}}(F)$,

$$\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}} : \mathrm{Hom}_{\mathcal{S}}(R[(F/H)^?], T) \to \mathrm{Hom}_{\mathcal{V}}(\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}R[(F/H)^?], \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}T)$$

is a bijection.

Surjectivity yields the existence of $R\mathcal{S}$-maps $t_n : \tilde{C}_n \to \tilde{C}_{n+1}$ such that $s_n = \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}t_n$. Injectivity then implies $\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}(t_{n-1}d_n + d_{n+1}t_n) = \mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}\mathrm{Id}_n$, and thus $t_{n-1}d_n + d_{n+1}t_n = \mathrm{Id}_n$, so $\tilde{C}_*(\Delta_S^?)$ is contractible in $CS_{\mathcal{S}}(F)$. Here we have also used the fact that $\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}$ is additive on $R[(F/H)^?]$; this is evident since Res is a forgetful functor. Finally, observe that the collection $\{s_n\}$ is a chain map if $\{t_n\}$ is, since the boundary maps in the complex $\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}\tilde{C}_*(\Delta_S^?)$ are by construction the images under $\mathrm{Res}_{\mathcal{V}}^{\mathcal{S}}$ of the boundary maps of $\tilde{C}_*(\Delta_S^?)$.

We now prove a lemma:

Lemma 3.8 *Let $0 \to E_* \to C_* \to P_* \to 0$ be a short exact sequence of chain complexes, where P_* is a complex of projectives and E_* is contractible. Then $C_* \cong P_* \oplus E_*$.*

Proof: Let $\alpha_n : E_n \to E_{n-1}$ and $\beta_n : P_n \to P_{n-1}$ be the differential maps on E_n and P_n, let $i_n : E_n \to E_n \oplus P_n$ and $j_n : P_n \to E_n \oplus P_n$ be the canonical inclusions, and let $q_n : E_n \oplus P_n \to E_n$ and $r_n : E_n \oplus P_n \to P_n$ be the canonical projections.

In each degree we have a short exact sequence

$$0 \to E_n \to C_n \to P_n \to 0$$

and since P_n is projective, we may assume for each n that $C_n \cong E_n \oplus P_n$ and the middle maps are i_n and r_n respectively. We should like the differential map d_n on C_n to be simply $\alpha_n \oplus \beta_n$, but in general this need not be true: the summands of C_n may be mapped into each other. However, it *is* true that d_n is well-behaved at least on the E_n summand: that is, given $x \in E_n$ and $0 = 0_{P_n}$, we have $d_n(x, 0) = d_n i_n(x) = i_{n-1}\alpha_n(x) = (\alpha_n(x), 0)$.

We also know, given $y \in P_n$, that the P_{n-1}-entry of $d_n(0, y)$ is $r_{n-1} d_n i_n(y) = \beta_n r_n i_n(y) = \beta_n(y)$. So the only difficulty is that P_n might map to E_{n-1} in a nontrivial way. This map $P_n \to E_{n-1}$ is the composition $\phi_n := q_{n-1} d_n j_n$, so that $d_n(x, y) = (\alpha_n(x) + \phi_n(y), \beta_n(y))$.

Now we define $\phi'_n = (-1)^n \phi_n$. We wish to show that the collection ϕ' of all the ϕ'_n is a chain map $P_* \to E_*$ of degree -1. We need to show $\alpha_{n-1}\phi'_n = \phi'_{n-1}\beta_n$, or equivalently $\alpha_{n-1}\phi_n = -\phi_{n-1}\beta_n$. Since i_{n-2} is a monomorphism, it is equivalent to show $i_{n-2}\alpha_{n-1}\phi_n = -i_{n-2}\phi_{n-1}\beta_n$. The proof is now a routine if slightly lengthy diagram chase; it uses the fact that inclusion followed by projection is the identity, that squares of boundary maps are zero, that the identity on a direct sum is the sum of the projections followed by the inclusions, and the chain map condition.

Thus ϕ' is a chain map of degree -1. Now the identity on E_* is nullhomotopic, so $\phi' = \mathrm{Id}_{E_*}\phi' \simeq 0\phi' = 0$ and ϕ' is nullhomotopic.

Thus we have maps $\theta_n : P_n \to E_n$ such that $\phi'_n = \theta_{n-1}\beta_n + \alpha_{n+1}\theta_n$. Now let $\theta'_n = (-1)^n \theta_n$, and define $\Theta : C_* \to E_* \oplus P_*$ by $\Theta(x, y) = (x + \theta'(y), y)$. Θ is then an isomorphism of complexes with $\Theta^{-1}(x, y) = (x - \theta'(y), y)$. To show that Θ and Θ^{-1} are chain maps amounts to showing that $\alpha\theta' = \phi + \theta'\beta$, or $\phi = \alpha\theta' - \theta'\beta$, and we know that $(-1)^n\phi_n = \phi'_n = \alpha_{n+1}\theta_n + \theta_{n-1}\beta_n$, so $\phi_n = \alpha_n\theta'_n - \theta'_{n-1}\beta_n$. This proves the lemma.

Thus $\tilde{C}_*(\Delta^?) \cong E_* \oplus P_*$ where $E_* = \tilde{C}_*(\Delta^?_S)$ is contractible and $P_* = C_*((\Delta, \Delta_S)^?)$ in $CS_{\mathcal{S}}(F)$ is a complex of projectives.

Proposition 3.9 *Evaluation of a coefficient system at a particular K (or F/K) is a functor $CS_{\mathcal{W}}(G) \to R[N_G(H)/H]$-mod, which: (1) respects direct sums, (2) sends projective objects to projective objects, and (3) sends contractible complexes to contractible complexes.*

Proof: Evaluation at K is a functor because the composition of natural transformations is defined by $(\theta\phi)_K = \theta_K\phi_K$, and it has values in $R[N_G(H)/H]$-mod as observed above. Let $ev_K : CS_{\mathcal{W}}(G) \to R$-mod be this functor.

To show (1) we could define the direct sum of two coefficient systems and show that it satisfies the universal property of the direct sum. This is straightforward. Alternatively, we can take direct sums in mod-$R\mathcal{W}$, apply the explicit equivalence of categories s: mod-$R\mathcal{W} \to CS_{\mathcal{W}}(G)$ from ([30], Theorem 7.1, [54], 2.1), and evaluate at K. We choose to use this second approach; the idea is that direct sums are very well-understood mod-$R\mathcal{W}$, while the evaluation functor is very well-understood in $CS_{\mathcal{W}}(G)$. We will also prove (2) and part of (3) this way.

Given $K \in \mathcal{W}$ and $L, M \in$ mod-$R\mathcal{W}$, the explicit formula in [30] and [54] tells us that $ev_K(s(L \oplus M)) = (s(L \oplus M)(K)$ is the R-module $(L \oplus M)\mathrm{Id}_K \cong (L\mathrm{Id}_K) \oplus M\mathrm{Id}_K)$, which is exactly $ev_K(s(L)) \oplus ev_K(s(M))$, as desired. Observe that we really get a *right* R-module structure this way, but for us R is commutative so left modules and right modules are the same.

(3) follows from (1), since a contractible complex (C_n, d_n) has the form $C_n \cong A_{n+1} \oplus B_n$ where $d_n|_{A_n} = 0$ and $d_n|_{B_n}$ is an isomorphism $B_n \to A_n$. Notice that ev_K sends zero maps to zero maps and therefore sends complexes to complexes, because the 0 object is sent to 0 by the same sort of argument as in the proof of (1).

To show (2), we note that projectives in both R-mod and mod-$R\mathcal{W}$ are summands of free modules. By (1), evaluation respects direct sums, so it is enough to show that evaluation sends free modules to free (or projective) modules; and by (1) again it is enough to show that $R\mathcal{W}$ is sent to a free R-module by evaluation at $K \in \mathcal{W}$.

As in (1), $s(R\mathcal{W})(K) = R\mathcal{W}\mathrm{Id}_K$. As an R-module, this is generated by all α such that $\alpha \circ \mathrm{Id}_K$ is defined; by definition of the category algebra $R\mathcal{W}$, it is therefore the free R-module on all α which have domain K. We want to show that this is in fact a free $R[N_G(K)/K]$-module. As seen above, $R[\{\alpha|\mathrm{Dom}(\alpha) = K] = R[\{i_{K^g \leq J} \circ c_g\}]$. $N_G(K)/K$ acts on the R-basis $\{i_{K^g \leq J} \circ c_g\}$ by $nK\dot{(}i_{K^g \leq J} \circ c_g) = i_{K^{ng} \leq J^n} \circ c_{ng} = i_{K^g \leq J^n} \circ c_{ng}$. This action is defined since $K^{ng} = (K^n)^g = K^g \leq J$. Thus $N_G(K)/K$ acts by permuting the symbols $c_g : K \to K^g$. This is a free action, which proves the proposition. $\square$

So, continuing with the proof of the Main Structure Theorem, we take our isomorphism $\tilde{C}_*(\Delta^?) \cong E_* \oplus P_*$ of complexes in $CS_{\mathcal{S}}(F)$, and evaluate at $K = 1$ to get an isomorphism of complexes of RF-modules: $\tilde{C}_*(\Delta) \cong P_*(1) \oplus E_*(1)$.

Now induce these complexes from F to G. In the previous paragraph we have considered $\tilde{C}_*(\Delta)$ as an RF-module. Properly, it is the RF-module $\mathrm{Res}_F^G \tilde{C}_*(\Delta)$ obtained by taking the RG-module $\tilde{C}_*(\Delta)$ and letting the subgroup F act. Thus we obtain $\mathrm{Ind}_F^G \mathrm{Res}_F^G \tilde{C}_*(\Delta) \cong \mathrm{Ind}_F^G E_*(1) \oplus \mathrm{Ind}_F^G P_*(1)$, and $P'_* := \mathrm{Ind}_F^G P_*(1)$ is projective and $E'_* := \mathrm{Ind}_F^G E_*(1)$ is contractible. Finally, $\tilde{C}_*(\Delta)$ is a summand of $\mathrm{Ind}_F^G \mathrm{Res}_F^G \tilde{C}_*(\Delta)$ by

$$c \mapsto \sum_{g \in G/F} g \otimes g^{-1} c, h \otimes c \mapsto |G : F|^{-1} hc, c \in \tilde{C}_*(\Delta), h \in G.$$

Therefore $\tilde{C}_*(\Delta)$ is a summand of $P'_* \oplus E'_*$. If Δ is a finite-dimensional CW-complex then the Krull-Schmidt theorem (see for instance [55], 11.5 and 11.6) guarantees that $\tilde{C}_*(\Delta) \cong Q_* \oplus D_*$ for some summands Q_* of P'_* and D_* of E'_*. Thus D_* is contractible and Q_* is a complex of projectives, finishing the proof of Webb's original theorem. However, in the new case where our complex is bounded below but not necessarily above, we need to convince ourselves that the Krull-Schmidt theorem still holds.

Definition 3.10 *([5], Definition A.1) An additive category $\mathfrak{C}$ is called:*

(i) ω-local *if every object $A \in \mathfrak{C}$ decomposes into a countable direct sum of objects with local endomorphism rings;*

(ii) fully additive *if any idempotent morphism in $\mathfrak{C}$ splits*

(iii) locally finite *(over $\boldsymbol{S}$) if it is fully additive and $Hom_{\mathfrak{C}}(A, B)$ is a finitely-generated $\boldsymbol{S}$-module for all objects $A, B \in \mathfrak{C}$.*

(iv) Krull-Schmidt *if every object has a* unique *decomposition into a direct sum of objects which have local endomorphism rings.*

Proposition 3.11 *([5], Proposition A.2) Suppose that $\boldsymbol{S}$ is a complete local noetherian ring. If $\mathfrak{C}$ is a locally-finite category over $\boldsymbol{S}$, then both the category $C(\mathfrak{C})$ of chain complexes of objects of $\mathfrak{C}$ and the homotopy category $K(\mathfrak{C})$ are ω-local.*

Proof: The proof appears in Appendix A of [5]. □

Theorem 3.12 *Let G be a finite group and R a complete p-local ring. The category $C(RG)$-mod of chain complexes of finitely-generated RG-modules is Krull-Schmidt.*

We apply Proposition 3.11, taking $\mathbf{S} = R$, and $\mathfrak{C}$ to be the category of finitely-generated RG-modules. We verify that the hypotheses of the above proposition are satisfied. Clearly R is a complete local noetherian ring: it is either a field or a complete discrete valuation ring, either of which is a complete local noetherian ring. We also need $\mathfrak{C}$ to be locally finite over R, which (by definition) means we need the following two facts:

(1) Every idempotent morphism $e : M \to M$ for M in $\mathfrak{C}$ splits, and (2) $\mathrm{Hom}_{\mathfrak{C}}(M, N)$ is finitely generated over R for all M, N.

Condition (1) follows immediately from Fitting's Lemma, which says (in particular) that for any $M \in \mathfrak{C} = RG$-mod, any endomorphism f of M must satisfy $M = \mathrm{Im}(f^m) \oplus \mathrm{Ker}(f^m)$ for some $m > 0$. If we also assume f is idempotent, then we see that f must split.

Condition (2) is clear if $R = k$ is a field since a finitely-generated kG-module is a finite-dimensional k-vector space as long as G is a finite group. More generally, suppose G is a finite group and R is a complete p-local ring, and let M and N be finitely-generated RG-modules. We want to show that $\mathrm{Hom}_{RG}(M, N)$ is a finitely-generated R-module.

First, since G is a finite group, M and N are finitely-generated over R. Then since R is a principal ideal domain, $M \cong R^r \oplus R/(a_1) \oplus \ldots \oplus R/(a_m)$ for some integers $r, m \geq 0$ and some $a_1, \ldots, a_m \in R$, and similarly for N. Thus we may write an R-homomorphism from M to N as an $(r+m) \times (r+m)$ matrix, so $\mathrm{Hom}_R(M, N)$ is finitely-generated over R.

Finally, R is a noetherian ring, so a finitely-generated R-module is a noetherian R-module. Thus $\mathrm{Hom}_R(M, N)$ is noetherian, so any R-submodule of $\mathrm{Hom}_R(M, N)$ is finitely-generated over R. But $\mathrm{Hom}_{RG}(M, N)$ is an R-submodule of $\mathrm{Hom}_R(M, N)$, so $\mathrm{Hom}_{RG}(M, N)$ is finitely-generated over R.

So if $\mathfrak{C} = RG$-mod then we have that that $C(\mathfrak{C})$ is ω-local, which is to say that every object in $C(\mathfrak{C})$ decomposes into a direct sum of objects which have local endomorphism rings. Of course, $C(\mathfrak{C})$ is not only additive but abelian (it has kernels, cokernels, and a 0 object). Moreover, $C(\mathfrak{C})$ is equivalent to a *small* category if and only if $\mathfrak{C}$ is. To see that $\mathfrak{C}$ is equivalent to a small category, we make two simple observations. First, $\mathrm{Hom}_{\mathfrak{C}}(A, B)$ is small for all A and B. Second, $\mathfrak{C}$ is equivalent to the category of all representations, whose objects are group homomorphisms $G \to GL(V)$ for some finitely-generated R-module V. Again, any finitely-generated V is a *quotient* of some R^n, and the collection of quotients V of R^n such that $n \geq 0$ is a set, as is then the collection of group homomorphisms $G \to GL(V)$ for all such V.

Therefore, $C(\mathfrak{C})$ is equivalent by the Freyd-Mitchell full imbedding theorem (see [29]) to a full subcategory of the category of T-modules for some ring T. Therefore by Corollary 4.1.5 of [13], any decomposition of an object of $C(\mathfrak{C})$ into objects with local endomorphism rings is unique up to isomorphism and reindexing. So in fact $C(\mathfrak{C})$ is Krull-Schmidt, proving the theorem. □

Corollary 3.13 *Suppose G is a finite group and R is a complete p-local ring. Let $P_* \oplus E_* = \bigoplus_{i \in I} M_*^i$ be a direct-sum decomposition of $P_* \oplus E_*$ into indecomposable complexes, where P_* is a complex of projective RG-modules, E_* is contractible, and P_* and E_* are both bounded below. Then each M_*^i is either a complex of projective modules or is contractible.*

Proof. By the Krull-Schmidt property, any indecomposable M_*^i in the decomposition is isomorphic to a summand either of P_* or of E_*. □

This finishes the proof of part (i) of Theorem 3.1.

Finally, any summand of any direct sum decomposition of an object of $C(\mathfrak{C})$ will also be an object in $C(\mathfrak{C})$, and will therefore (again by Proposition A.2 in [5]!) have a decomposition into objects with local endomorphism rings. So in fact any indecomposable summand must have a local endomorphism ring. (The converse is easy; a decomposable summand has a non-local endomorphism ring, since we may write the identity as the sum of two non-invertible projection maps). Therefore every object in $C(\mathfrak{C})$ decomposes uniquely into indecomposables. This finishes part (ii) of Theorem 3.1. □

Remark. The reader is invited to see [32] for an alternative treatment of the Krull-Schmidt theorem in arbitrary categories.

Remark. Lemma 3.8, which is used in proving part (i) of Theorem 3.1, is a modification of a lemma used by Symonds in [47]. In our lemma, the complexes may be unbounded. In Symonds' lemma, the complexes must be bounded *below*. Both lemmas therefore apply to the chain complex of a finite- or infinite-dimensional CW-complex.

Remark. Symonds has adapted this proof to prove a stronger claim, at least in the case of a bounded CW-complex:

Theorem 3.14 *([45]) Let p be a prime, R a complete p-local ring, G a finite group, and Δ a bounded G-CW-complex. Suppose that the fixed point set Δ^P is R-acyclic for each p-subgroup $P \leq G$ that intersects H non-trivially. Let $\tilde{C}_*(\Delta)$ denote the augmented CW-chain complex of Δ over R, considered as a complex of RG-modules.*

Then $\tilde{C}_*(\Delta) \cong P_* \oplus E_*$, where P_* is a complex of trivial source RG-modules that are projective relative to subgroups that have trivial intersection with H, and E_* is split exact.

Remark. Webb's original theorem then follows by taking $H = G$.

Corollary 3.15 *([45]) Let Γ be a finite group and let $\Delta = |\mathcal{S}_p(\Gamma)|$. Thus $Aut(\Gamma)$ acts on Δ and also on $\tilde{C}_*(\Delta)$.*

Then $\tilde{C}_*(\Delta) \cong P_* \oplus E_*$ as a complex of $RAut(\Gamma)$-modules, where P_* is a complex of $RAut(\Gamma)$-modules which are projective on restriction to Γ (via the map $\Gamma \to Inn(\Gamma) \leq Aut(\Gamma)$) and E_* is split exact.

The Corollary ultimately comes from applying the Theorem with $G = \mathrm{Aut}(\Gamma)$ and $H = \mathrm{Inn}(\Gamma)$. We do not use or prove these statements here, but the reader is invited to see [45] for details.

Chapter 4

Alternative complexes

Let G be a finite group, and p a prime.

So far we have been working with the G-CW-complexes $|\mathcal{S}_p(G)|$, $|\mathcal{A}_p(G)|$, *etc.*, which are all G-homotopy-equivalent and which are all bounded. The reason for looking at these complexes in the first place was that they generalized the building of a finite group of Lie type in characteristic p. More precisely, if G is a finite group of Lie type in characteristic p, then these complexes have homology only in one degree, where the homology is the usual Steinberg module (Steinberg representation).

Bill Dwyer ([16], [17]) has found various other constructions of G-CW-complexes, which all have the same homology as the subgroup complexes we have been considering, but which may be unbounded. In fact Dwyer exhibits homology isomorphisms from his infinite-dimensional complexes to $|\mathcal{S}_p(G)|$. These are constructed as the nerves of various G-categories.

We should like to know whether or not Dwyer's "alternative" complexes are of the same homotopy type as $\mathcal{S}_p(G)$. (Recall that we know that $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, *etc.*, all have the same homotopy type.) If not, then these new unbounded complexes would be new candidates for the title of "Steinberg complex."

We now state the following general theorem.

Theorem 4.1 *Let A_* and B_* be complexes of projective modules which are bounded below, and suppose ϕ is a chain map $A_* \to B_*$ which induces isomorphism on homology. Then ϕ is a homotopy equivalence.*

Proof: Let $C(\phi)$ be the mapping cone of ϕ, so $C(\phi)_n = A_{n-1} \oplus B_n$, with differential $d_n : C(\phi)_n \to C(\phi)_{n-1}$ given by $(a_{n-1}, b_n) \mapsto (-d^A_{n-1}(a_{n-1}), \phi_{n-1}(a_{n-1}) + d^B_n(b_n))$.

We thus have a short exact sequence of complexes,

$$0 \to B_* \to C(\phi) \to A_*[-1] \to 0,$$

where $A_*[-1]$ is the chain complex with entry A_{n-1} in degree n, together with the obvious differential maps. Then $H_{n+1}(A_*[-1]) = H_n(A_*)$, and therefore we have a long exact sequence:

$$\ldots \to H_{n+1}(C(\phi)) \to H_n(A_*) \to H_n B_* \to H_n(C(\phi)) \to \ldots,$$

and in fact the map $H_n(A_*) \to H_n(B_*)$ is $H_n(\phi)$. (This last statement is Lemma 1.5.3 of [57].) Now since ϕ induces isomorphism on homology, we see that $H_n(C(\phi)) = 0$ for all n. On the other hand, $C(\phi)_n = A_{n-1} \oplus B_n$ is projective for all n, so $C(\phi)$ is a complex of projectives which is acyclic and bounded below. Therefore $C(\phi)$ is contractible. But $C(\phi)$ is contractible if and only if ϕ is a homotopy equivalence (see for instance Proposition 2.8 in Chapter 3 of [3]). □

Theorem 4.2 *If J_* is the (unique up to isomorphism) minimal summand of projectives coming from any of Dwyer's complexes, then it is homotopy-equivalent to the usual Steinberg complex.*

Proof: From each of his complexes, Dwyer exhibits a map to the p-subgroups complex which induces homology isomorphism. (This map is induced by a functor from the category whose nerve he takes for his complex.)

Let R be a complete p-local ring. Then upon composition with the RG-chain homotopy equivalences $J_* \simeq X^{\beta}_{\mathcal{C}}$ (this equivalence exists by part (i) of the Main Structure Theorem) and $\tilde{C}(\mathcal{S}_p(G); R) \simeq \mathrm{St}_*$, we get a map $\phi : J_* \to \mathrm{St}_*$ between bounded-below complexes of projectives, which induces isomorphism on homology. Thus J_* and St_* are RG-chain homotopy-equivalent. □

Chapter 5

Explicit calculations of a Steinberg complex, part I

The next several sections are devoted to examining a particular Steinberg complex in great detail. It does seem to be quite challenging to calculate Steinberg complexes in general. Still, we should like to have some concrete examples at our disposal, and this type of exploration has not appeared much in the literature on the topic to date. The question becomes, which example should we try first?

The Steinberg complex is a complex of projective kG-modules, and one may ask whether or not its homology groups must also be projective as kG-modules. It is now known that the homology of the Steinberg complex need not always be projective. A few examples of groups whose Steinberg complexes will exhibit non-projective homology may be found in [49]. It is one of these examples that we should like to examine first.

Let C_n be the cyclic group of order n, and let $G_{21} = C_7 \rtimes C_3$ be the non-abelian group of order 21. G_{21} may be realized as the subgroup of S_7 generated by (1234567) and (235)(476), since

$$(1234567)^2 = (1357246) = (235)(476)(1234567)(253)(467).$$

Then let $W = G_{21} \wr C_3 = ((C_7 \rtimes C_3) \times (C_7 \rtimes C_3) \times (C_7 \rtimes C_3)) \rtimes C_3$, where the complement C_3 acts by permuting the copies of G_{21} cyclically. Thus

$$W = \langle (1234567), (235)(476),$$

$$(1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21)\rangle,$$

and W has order $3^4 7^3 = 27783$. We wish to determine the Steinberg complex for the finite group W, at the prime $p = 3$, over the ring $R = \mathbb{F}_3$.

We start by finding $\mathcal{B}_3(W)$, the W-poset of nonidentity 3-subgroups $H \leq W$ satisfying $O_3(N_W(H)) = H$. (Recall that for q a prime and G a finite group, $O_q(G)$ is the unique largest normal q-subgroup of a finite group, equal to the intersection of all q-Sylow subgroups of G.)

There are only four W-conjugacy classes of such subgroups of W. To enumerate these conjugacy classes we can restrict our attention to subgroups of a single Sylow 3-subgroup of W, since the Sylow 3-subgroups are all W-conjugate by Sylow's Theorem. Let

$$S = ((C_3 \times 0) \times (C_3 \times 0) \times (C_3 \times 0)) \rtimes C_3, \text{ or}$$

$$S = \langle (2,3,5)(4,7,6), (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21)\rangle.$$

Then S has order 3^4 so $S \in \mathrm{Syl}_3(W)$. When it is clear that we are working inside S we may identify $((C_3 \times 0) \times (C_3 \times 0) \times (C_3 \times 0)) \rtimes C_3$ with $(C_3 \times C_3 \times C_3) \rtimes C_3$. We also let C denote the complement $0 \times 0 \times 0 \times C_3$.

Lemma 5.1 *Let $H \leq S$. If $(a_1, a_2, a_3, 0) \in H$ and $a_i \neq 0$ for all i, then $N_W(H) = N_S(H)$.*

Proof: S consists of exactly the elements of W which have 0 in all C_7-components. So we consider an arbitrary element normalizing H and show that it has no nonzero C_7-component.

Writing W as $((C_7 \rtimes C_3) \times (C_7 \rtimes C_3) \times (C_7 \rtimes C_3)) \rtimes C_3$, we conjugate the element $h = (0, a_1, 0, a_2, 0, a_3, 0) \in H \leq S$ by $t = (x_1, y_1, x_2, y_2, x_3, y_3, z)$. Then if $\phi : C_3 \to \mathrm{Aut}(C_7)$ is the homomorphism giving the semidirect product $C_7 \rtimes C_3$ described above, we find that the i^{th} G_{21}-component of $t * h * t^{-1}$ is (operations in subscripts are carried out mod 3):

$$\begin{aligned}(x_i,y_i)*(0,a_{i-z})*(x_i,y_i)^{-1} &= (x_i+\phi(y_i)(0), y_i+a_{i-z})*(\phi(-y_i)(-x_i), -y_i)\\ &= (x_i+\phi(y_i+a_{i-z})(\phi(-y_i)(-x_i)), y_i+a_{i-z}-y_i)\\ &= (x_i+(\phi(y_i+a_{i-z})\circ\phi(-y_i))(-x_i), a_{i-z})\\ &= (x_i+\phi(y_i+a_{i-z}-y_i)(-x_i), a_{i-z})\\ &= (x_i+\phi(a_{i-z})(-x_i), a_{i-z})\end{aligned}$$

So if $t*h*t^{-1} \in H \leq S$, then $-x_i = \phi(a_{i-z})(-x_i)$, which is true if and only if $a_{i-z} = 0$ or $x_i = 0$. Since $a_{i-z} \neq 0$ by assumption, this yields $x_i = 0$, and $N_W(H) \subseteq S$. $\square$

Corollary 5.2 *Let $H \leq S$. If $(a_1, a_2, a_3, 0) \in H$ and $i \neq 0$ for all i, then*

$$O_3(N_W(H)) = N_S(H).$$

Proof: By Lemma 1.1, $N_W(H) = N_S(H)$, so $N_W(H) \leq S$. Therefore $N_W(H) = N_S(H)$ is a 3-group, and equal to its 3-core. $\square$

Proposition 5.3 *Any nonidentity subgroup $H \leq S$ satisfying $O_3(N_W(H)) = H$ is W-conjugate either to: (i) S, (ii) C, or (iii) a subgroup of $C_3 \times C_3 \times C_3 \times 0$.*

Proof: Write C_3 additively as $\{0, 1, 2\}$, and write $S = C_3 \times C_3 \times C_3 \rtimes C_3$. Now let $H \leq S$ such that $O_3(N_W(H)) = H$, and assume that H is not a subgroup of $C_3 \times C_3 \times C_3 \times 0$. That is, the restriction to H of the canonical projection $\pi : S \to C$ is surjective. If this map $H \to C$ is an isomorphism, then $H = \langle h \rangle$ is cyclic of order 3. Then the first three components (the non-C components) of h sum to 0. In this case, $H = \langle h \rangle$ is conjugate to C: assume without loss of generality that h has C-component equal to 1, and write $h = (a, b, -a-b, 1)$. Then

$$(a, 0, -b, 1)(a, b, -a-b, 1)(0, b, -a, 2) = (a-a-b, 0+a, -b+b, 2)(0, b, -a, 2)$$

$$= (-b, a, 0, 2)(0, b, -a, 2) = (-b+b, a-a, 0+0, 1) = (0, 0, 0, 1),$$

so h is conjugate to the generator of C, and $\langle h \rangle$ is conjugate to C.

So we assume that $H \to C$ has a nontrivial kernel, K. K must have order 3,9, or 27. If $|K| = 27$, then $H = S$ and we are done.

First suppose $|K| = 3$, so that $|H| = 9$ and H is abelian. In particular every element of H centralizes every element of K. We know we have nonidentity elements $k, h \in H$ such that $\pi(k) = 0, \pi(h) = 1$, so $k = (a, b, c, 0)$ and $h = (t, u, v, 1)$. Since h centralizes k, we calculate that $k = (a, a, a, 0)$ with $a \in \{1, 2\}$. From this, Corollary 5.2 yields that $O_3(N_W(H)) = N_S(H)$.

Now, either $H \cong C_9$ or $H \cong C_3 \times C_3$. If H is not cyclic, then H is generated by $k = (a, a, a, 0)$ and $h = (t, u, v, 1)$, both of order 3. Thus $h = (t, t, t, 1)$. Hence H is generated by $(1, 1, 1, 0)$ and $(0, 0, 0, 1)$, so $H = \Delta(C_3) \times C_3$, where $\Delta(C_3)$ is the diagonally-embedded subgroup $\langle (1, 1, 1) \rangle$ of $C_3 \times C_3 \times C_3$. But now direct calculation on the generators shows that $(1, 2, 0, 0) \in N_S(H) - H$, and therefore $H \neq N_S(H) = O_3(N_W(H))$.

On the other hand, suppose that H is cyclic, so $H \cong C_9$. In this case $H = \langle (t, u, v, 1) \rangle$, with $t + u + v \neq 0$ (again, $(1, 1, 1, 0) \in K \leq H$). Once again it turns out that we have $(1, 2, 0, 0) \in N_S(H) - H$ so once again $O_3(N_W(H)) \neq H$.

So finally we assume that $|K| = 9$, and $|H| = 27$. In particular, H is a maximal subgroup of S. This shows (see 5.2.4 in [34], p.130) that $H \triangleleft S$, so $O_3(N_S(H)) = N_S(H) = S \neq H$. We wish to show, therefore, that $N_W(H) = N_S(H)$. We should like to show $(1, 1, 1, 0) \in H$, since then Lemma 1.1 would finish the proof. For this, it suffices to show that $(1, 1, 1, 0)$ is an element of the Frattini subgroup $F = \mathrm{Frat}(S)$, since F is the intersection of all maximal subgroups of S.

Since S is a 3-group, the Burnside Basis Theorem ([34], 5.3.2, p.140) states that $F = S'S^3$. So we need only exhibit a factorization of $(1, 1, 1, 0)$ as the product of an element of S' and an element of S^3. But we already know that, for instance, $(1, 1, 1, 0) = (1, 0, 0, 1)^3$. $\square$

Theorem 5.4 *The nonidentity subgroups $H \leq W$ which satisfies $O_3(N_W(H)) = H$ are exactly the W-conjugates of: (i) S, (ii) C, (iii) $C_3 \times 0 \times 0 \times 0$, and (iv) $C_3 \times C_3 \times 0 \times 0$.*

Proof: The 3-subgroups $H \leq W$ are W-conjugate to subgroups of S, by Sylow's Theorem, so the only candidates are those listed in Proposition 5.3. By Corollary 5.2,

$O_3(N_W(S)) = S$ (and by Lemma 1.1, $N_W(S) = N_S(S) = S$; later we shall want this fact as well). To show (ii), we let $x = (0,0,0,0,0,0,1) \in C$, so $C = \langle x \rangle$. Then

$$N_W(C) = \{a \in W | a^{-1}xa \in \{x, x^2\}\}$$

However, given $a = (x_1, y_1, x_2, y_2, x_3, y_3, z) \in W$, we have $a^{-1}xa = x^2 \Leftrightarrow xa = ax^2 \Leftrightarrow (0,0,0,0,0,0,1)(x_1, y_1, x_2, y_2, x_3, y_3, z) = (x_1, y_1, x_2, y_2, x_3, y_3, z)(0,0,0,0,0,0,2) \Rightarrow 1 + z = z + 2$, which is impossible. Thus

$$N_W(C) = \{a \in W | a^{-1}xa = x\}$$

But let $a \in W$. Then $a^{-1}xa = x \Leftrightarrow xa = ax \Leftrightarrow xax^{-1} = a \Leftrightarrow a \in \Delta(G_{21}) \times C_3$, so $N_W(C) = \Delta(G_{21}) \times C_3$.

Now, C is certainly a normal 3-subgroup of $N_W(C)$, of order 3. The Sylow 3-subgroups of $\Delta(G_{21}) \times C_3$ have order 9, but there are more than one of these: there is one for each of the Sylow 3-subgroups of G_{21}, and there are exactly 7 of these. Thus the Sylow 3-subgroups of $N_W(C)$ are not normal, and so the largest normal 3-subgroup of $N_W(C)$ is C, verifying (ii).

It remains to show that every nonidentity subgroup $H \leq C_3 \times C_3 \times C_3 \times 0$ satisfying $O_3(N_W(H)) = H$ is W-conjugate either to $C_3 \times 0 \times 0 \times 0$ or to $C_3 \times C_3 \times 0 \times 0$. Let D denote $C_3 \times C_3 \times C_3$. By Corollary 5.2, $D \triangleleft S$ implies $O_3(N_W(D)) = S \neq D$. So a nonidentity subgroup $H \leq D$ satisfying $O_3(N_W(H)) = H$ must have order either 3 or 9.

There are $27 - 1 = 26$ elements of D of order 3, so there are $26/2 = 13$ subgroups of D of order 3. We can also count the subgroups of D of order 9: If we wish to construct such a subgroup, we can choose a generator in any of $27 - 1 = 26$ ways, then choose another generator in any of $27 - 3 = 24$ ways to get a subgroup of order 9. But given the first generator, $9 - 3 = 6$ distinct choices for the second generator will yield the same subgroup. This means that every nonidentity element of D is contained in exactly $24/6 = 4$ subgroups of D of order 9. Now the total number of subgroups of order 9 is the number of nonidentity elements of D, times the number of order-9 subgroups containing a given nonidentity element, divided by the number of nonidentity elements in each order-9 subgroup, or $26 * 4/8 = 13$. So there are 13 order-3 subgroups and 13 order-9 subgroups. (See also [36], where the identity

$$\sum_{\lambda=0}^{m}(-1)^{\lambda}p^{\binom{\lambda}{2}}\mathcal{E}_{\lambda}(G)=0$$

is proven for G an arbitrary p-group: here $|G|=p^m$, and $\mathcal{E}_\lambda(G)$ denotes the number of elementary abelian subgroups of G of size p^λ. This work is also cited in [37].)

Omitting parentheses and commas, the order-9 subgroups are generated by

$$(110,001),(010,001),(100,001),(210,001),(100,021),(100,010),(010,201),$$

$$(010,101),(110,201),(100,011),(210,201),(210,101),\text{ and }(110,101).$$

Except for $\langle 010,001\rangle,\langle 100,001\rangle$, and $\langle 100,010\rangle$, all of these subgroups satisfy the hypotheses of Corollary 5.2, so for these groups the fact that D is abelian yields $O_3(N_W(H)) = N_S(H) \geq D \gneq H$. The remaining order-9 subgroups are visibly W-conjugate (actually S-conjugate) to $\langle 100,010\rangle = C_3\times C_3\times 0$. Let H denote this subgroup. We have $D\leq N_W(H)$, and we see that $0\times 0\times G_{21}\times 0\leq N_W(H)$, so $C_3\times C_3\times G_{21}\times 0\leq N_W(H)$. Following the calculation in the proof of Lemma 1.1, and letting $(a_1,a_2,a_3)=(1,1,0)\in H$, we see that if $t=(x_1,\dots,y_3,z)\in W$ normalizes H, then for each i, either $x_i=0$ or $a_{i-z}=0$. So taking $z=0$ for the moment, we see that $N_W(H)\bigcap G_{21}\times G_{21}\times G_{21}=C_3\times C_3\times G_{21}$. Suppose on the other hand that $z=1$, so that $x_2=0,x_3=0$ (since $a_{2-1}\neq 0, a_{3-1}\neq 0$). Then

$$\begin{aligned}
N_W(H)\ni t^2 &= (x_1,y_1,0,y_2,0,y_3,1)^2\\
&= ((x_1,y_1)*(0,y_3),(0,y_2)*(x_1,y_1),(0,y_3)*(0,y_2),1+1)\\
&= ((x_1+\phi(y_1)(0),y_1+y_3),(0+\phi(y_2)(x_1),y_2+y_1),\\
&\qquad (0+\phi(y_3)(0),y_3+y_2),2)\\
&= ((x_1,y_1+y_3),(\phi(y_2)(x_1),y_2+y_1),(0,y_3+y_2),2).
\end{aligned}$$

This time, since $a_{3-2}\neq 0, a_{1-2}\neq 0$, the first C_7-coordinate of t^2 must be zero. But that says exactly that $x_1=0$, so we see that $x_i=0$ for all i. Hence any element of $N_W(H)$ with nonzero C-coordinate must be an element of S. But we know that $D\leq N_W(H)$, so if there is any element $t\in N_W(H)$ with nonzero C-coordinate, then $C\leq N_W(H)$.

But we can see that C does not normalize H. Thus $N_W(H) = C_3 \times C_3 \times G_{21}$. Finally $O_3(N_W(H))$ must have order at most 9, since $N_W(H)$ has non-normal (non-unique) Sylow 3-subgroups of order 27, but H is a normal 3-subgroup of its normalizer, of order 9. Thus $O_3(N_W(H)) = H$, as claimed.

We are now reduced to showing that the order-3 subgroups of D which belong to $\mathcal{B}_3(W)$ are W-conjugate to $C_3 \times 0 \times 0$. The order-3 subgroups are generated, respectively, by

$$100, 010, 110, 120, 001, 101, 201, 011, 012, 111, 112, 121, \text{ and } 211.$$

Of these, those generated by $111, 112, 121,$ and 211 have, by Corollary 1.2, $O_3(N_W(H)) = N_S(H) \geq D \gneqq H$, since D is abelian. The remaining subgroups of D visibly fall into at most 3 W-conjugacy classes. We consider a subgroup from each conjugacy class.

First let $H = \langle(1,1,0)\rangle \leq D$. We copy the work we did in showing that $N_W(C_3 \times C_3 \times 0) \leq C_3 \times C_3 \times G_{21}$ to show that $N_W(H) \leq C_3 \times C_3 \times G_{21}$, since to eliminate other elements we only considered their effect on the element $(1,1,0)$. Direct calculation then shows that $N_W(H) = C_3 \times C_3 \times G21$. But we saw above that $O_3(C_3 \times C_3 \times G_{21} = C_3 \times C_3 \times 0 \neq H$. The same work also suffices to show that $N_W(\langle(1,2,0)\rangle) = C_3 \times C_3 \times 0 \neq (\langle(1,2,0)\rangle$, since all we used about the element $(1,1,0)$ was that it was nonzero in the first two components.

Finally let $H = C_3 \times 0 \times 0 = \langle(1,0,0)\rangle \leq D$. We claim that $N_W(H) = C_3 \times G_{21} \times G_{21} \times 0$. First we assume for contradiction that $t = (x_1, \ldots, y_3, 1) \in N_W(H)$. We calculate that $t^{-1} = ((x_2, y_2)^{-1}, (x_3, y_3)^{-1}, (x_1, y_1)^{-1}), 2)$, so the first G_{21}-component of $t * (0,1,0,0,0,0,0) * t^{-1}$ is $(x_1, y_1) * (x_1, y_1)^{-1} = (0,0)$. Since we have assumed that $t \in N_W(H)$, we see that conjugation by t sends a generator of H to zero, a contradiction. So $N_W(H) \leq G_{21} \times G_{21} \times G_{21}$. We can see directly that $C_3 \times G_{21} \times G_{21} \leq N_W(H)$, so $N_W(H)$ is either $G_{21} \times G_{21} \times G_{21}$ or $C_3 \times G_{21} \times G_{21}$. But we see that a generator of $C_7 \times 0 \times 0$ does not normalize H. Thus $N_W(H) = C_3 \times G_{21} \times G_{21}$. However,

$$\begin{aligned} O_3(N_W(H)) &= \bigcap_{P \in \mathrm{Syl}_3(N_W(H))} P \\ &= C_3 \times (\bigcap_{Q \in \mathrm{Syl}_3(G_{21})} Q) \times (\bigcap_{R \in \mathrm{Syl}_3(G_{21})} R) \\ &= C_3 \times 0 \times 0. \end{aligned}$$

$\square$

Chapter 6

Explicit calculations of a Steinberg complex, part II

As in the last section, let $W = G_{21} \wr C_3$, where G_{21} is up to isomorphism the unique non-abelian group of order 21, and let $p = 3$. We continue to find information about the Steinberg complex of W at $p = 3$ over the ring $R = \mathbb{F}_3$.

Let us choose one representative from each of the four conjugacy classes of subgroups appearing in $\mathcal{B}_3(W)$ and give them all names: We have $S = C_3 \times C_3 \times C_3 \rtimes C_3$ and $C = 0 \times 0 \times 0 \times C_3$. Now let A denote $C_3 \times C_3 \times 0 \times 0$ and B denote $C_3 \times 0 \times 0 \times 0$.

We have chosen our representatives in such a way that $B \subseteq A \subseteq S$ and $C \subseteq S$. In fact this is the full list of inclusions possible:

Proposition 6.1 *No conjugate of C is a subgroup of any conjugate of A.*

This says that no other set of representatives would have yielded a larger set of inclusions of subgroups, since by cardinality this is the only candidate for an additional inclusion.

Proof: Suppose ${}^gC \subseteq {}^hA$ for some $g, h \in W$; this would yield ${}^xC \subseteq A$ for some $x \in W$. But this is impossible; direct computation shows that the C-component of an element of W is never changed by conjugation. $\square$

In particular, C and B are not conjugate, so we have a minimal list of representatives of $\mathcal{B}_3$.

Proposition 6.2 *The W-stabilizer of a simplex $\sigma = (H_0 \leq \ldots \leq H_n)$ is $\bigcap_{i=0}^{n} N_W(H_i)$.*

Proof: By induction on n. The stabilizer of a vertex H_0 is $\{g \in W | gHg^{-1} = H\} = N_W(H)$. If $n \geq 1$, then the stabilizer of $H_0 \leq \ldots \leq H_n$ is $\{g \in W | gH_ig^{-1} = H_i \forall i\} = \{g \in W | gH_ig^{-1} = H_i, 0 \leq i \leq n-1\} \cap \{g \in W | gH_ng^{-1} = H_n\} = (\bigcap_{i=0}^{n-1} N_W(H_i)) \cap N_W(H_n) = (\bigcap_{i=0}^{n} N_W(H_i))$. $\square$

We have in the previous section computed the normalizers of the various subgroups (up to conjugacy) in the G-poset $\mathcal{B}_3(W)$. This together with the last proposition allows us to compute the stabilizer of each of our simplices:

$$\begin{aligned}
N_W(S) &= S \\
N_W(C) &= \Delta(G_{21}) \times C_3 \\
N_W(B) &= C_3 \times G_{21} \times G_{21} \times 0 \\
N_W(A) &= C_3 \times C_3 \times G_{21} \times 0 \\
\mathrm{Stab}_W(C \leq S) &= \Delta(C_3) \times C_3 \\
\mathrm{Stab}_W(B \leq A) &= C_3 \times C_3 \times G_{21} \times 0 \\
\mathrm{Stab}_W(B \leq S) &= C_3 \times C_3 \times C_3 \times 0 \\
\mathrm{Stab}_W(A \leq S) &= C_3 \times C_3 \times C_3 \times 0 \\
\mathrm{Stab}_W(B \leq A \leq S) &= C_3 \times C_3 \times C_3
\end{aligned}$$

We now calculate the size of the W-orbits of the simplices in our list, using the fact that $|W\sigma| = |W : \mathrm{Stab}_W(\sigma)|$:

$$
\begin{aligned}
|W : \mathrm{Stab}_W(S)| &= 7^3 \\
|W : \mathrm{Stab}_W(C)| &= 3^2 * 7^2 \\
|W : \mathrm{Stab}_W(B)| &= 3 * 7 \\
|W : \mathrm{Stab}_W(A)| &= 3 * 7^2 \\
|W : \mathrm{Stab}_W(C \leq S)| &= 3^2 * 7^3 \\
|W : \mathrm{Stab}_W(B \leq A)| &= 3 * 7^2 \\
|W : \mathrm{Stab}_W(B \leq S)| &= 3 * 7^3 \\
|W : \mathrm{Stab}_W(A \leq S)| &= 3 * 7^3 \\
|W : \mathrm{Stab}_W(B \leq A \leq S)| &= 3 * 7^3
\end{aligned}
$$

We saw in the previous section that S includes exactly three subgroups conjugate to A, and three subgroups conjugate to B. Each A contains two of the subgroups conjugate to B, and there are $7^3 = 343$ subgroups conjugate to S. Thus there are $7^3 * 3 * 2 = 2058$ 2-simplices in $|\mathcal{B}_3(W)|$. Each orbit of 2-simplices has size $3 * 7^3$, so there are exactly 2 orbits of 2-simplices. In fact there are 2 orbits of simplices of the form ${}^yB \leq {}^xA$, since no element will simultaneously send $C_3 \times 0 \times 0 \times 0$ to $0 \times C_3 \times 0 \times 0$ and $C_3 \times C_3 \times 0 \times 0$ to $C_3 \times C_3 \times 0 \times 0$.

Continuing in this way, S contains 3 subgroups conjugate to A, so there are $3 * 7^3$ 1-simplices of the form ${}^yA \leq {}^xS$, and the orbit of $S \leq A$ has size $3 * 7^3$, so there is only one orbit of 1-simplices of this form. Similarly there is only one orbit of simplices of the form ${}^yB \leq {}^xS$. It remains to determine the number of orbits of simplices of the form ${}^yC \leq {}^xS$.

As we stated before, any conjugate of C will have a nontrivial C-component. Any subgroup of S of order 3 that has nontrivial C-component must have generator $(t, u, v, 1)$ with $t+u+v = 0$. There are $3*3*1 = 9$ choices for such triples $(t, u, v) = (t, u, -t-u)$. All of these give subgroups of S conjugate to C: given $(t, u, -t-u, 1) \in S$, $(t, u, -t-u, 1) = (0, u, -t, 0)(0, 0, 0, 1)(0, -u, t, 0)$. So there are $3^2 * 7^3$ 1-simplices of the form ${}^xC \leq {}^yS$, and every orbit of such simplices has size $3^2 * 7^3$. So there is exactly one such orbit.

We can also say, therefore, that each conjugate of C is contained in exactly $|W :$

$\mathrm{Stab}_W(C \leq S)|/|W : \mathrm{Stab}_W(C)| = 3^2 * 7^3/(3^2 * 7^2) = 7$ conjugates of S. Similarly each conjugate of B is contained in $3 * 7^3/(3 * 7) = 7^2$ conjugates of S and is contained in 14 conjugates of A.

The orbit complex $|\mathcal{B}_3(W)|/W$ thus has 4 vertices (S, C, B, A), 5 1-simplices $(C \leq S, B \leq S, A \leq S, B \leq A, {}^cB \leq A)$, and 2 2-simplices $(B \leq A \leq S, {}^cB \leq A \leq S)$. The orbit complex is homotopy equivalent to the cone over the wedge of a 0-sphere and a 1-sphere, which is contractible.

In fact it is *always* true that the orbit complex will be contractible:

Theorem 6.3 *(Webb's Conjecture.) Given any finite group G and prime p dividing $|G|$, the orbit complex $|\mathcal{S}_p(G)|/G$ (or $|\mathcal{A}_p(G)|/G$, $|\mathcal{B}_p(G)|/G$,* etc.*) is contractible.*

Proof: See [45].

Remark: Suppose G is a group whose order is divisible by a prime q. Then it is clear that every Sylow q-subgroup of G lies in $\mathcal{B}_q(G)$ If the Sylow q-subgroups are the *only* elements of $\mathcal{B}_q(G)$, then $|\mathcal{B}_q(G)|/G = Q$ is a single vertex since all the Sylow p-subgroups are conjugate. On the other hand, suppose that $\mathrm{Syl}_q(G) \subsetneq \mathcal{B}_q(G)$. One might think, since every q-subgroup of G is contained in some Sylow q-subgroup and the Sylow q-subgroups are all conjugate, that $|\mathcal{B}_q(G)|/G$ must always be the cone over the space $|\mathcal{B}_q(G) - \mathrm{Syl}_q(G)|/G$. This would immediately prove Webb's conjecture, but in fact this argument will fail in general. As we have already seen in the example of $\mathcal{B}_3(W)$ at hand, a pair of vertices in the orbit complex may be connected by multiple edges. In particular, if Q is the vertex of the orbit complex which represents the Sylow q-subgroups of G, then Q and another vertex may be connected in the orbit complex by multiple edges, which cannot happen in the cone over $|\mathcal{B}_q(G) - \mathrm{Syl}_q(G)|$. This situation occurs if there is a Sylow q-subgroup Q with two subgroups $X, Y \in \mathcal{B}_q(G)$ such that X is G-conjugate to Y but not $N_G(Q)$-conjugate to Y.

Chapter 7

Explicit calculations of a Steinberg complex, part III

As in the previous sections, let $W = G_{21} \wr C_3$, where G_{21} is up to isomorphism the unique non-abelian group of order 21, and let $p = 3$. We continue to find information about the Steinberg complex of W at $p = 3$ over the ring $R = \mathbb{F}_3$.

Let $\Delta = |\mathcal{B}_3(W)|$ and let $k = \mathbb{F}_3$. As a complex of vector spaces over k, we calculated in the last section that the chain complex $\tilde{C}_*(\Delta; k)$ is

$$0 \to \left(k^{3*7^3} \oplus k^{3*7^3}\right) \to \left(k^{3*7^3} \oplus k^{3*7^3} \oplus k^{3*7^2} \oplus k^{3*7^2} \oplus k^{3^2*7^3}\right)$$

$$\to \left(k^{7^3} \oplus k^{3^2*7^2} \oplus k^{3*7} \oplus k^{3*7^2}\right) \to k \to 0,$$

where the nonzero terms are in dimensions 2, 1, 0, and -1.

Now let U be a small open neighborhood of M in Δ, where M the subspace of Δ obtained by removing all vertices in the W-orbit of the vertex C, and removing all edges in the orbit of the edge $C \leq S$. Let V be a small neighborhood of the subspace of Δ consisting only of the vertices in the orbits of C and S and the edges in the orbit of $C \leq S$. Then $U \cup V = \Delta$, and $U \cap V$ is homotopy-equivalent to the discrete space with one vertex for each conjugate of S. This discrete space may be described as $\bigvee_{i=1}^{342} S^0$, the bouquet of 342 copies of the 0-sphere, since the 0-sphere is a discrete space with two points. Thus:

Proposition 7.1 *We have a Mayer-Vietoris exact sequence:*

$$0 \to \tilde{H}_2(U \cap V) \to \tilde{H}_2(U) \oplus \tilde{H}_2(V) \to \tilde{H}_2(\Delta)$$

$$\to \tilde{H}_1(U \cap V) \to \tilde{H}_1(U) \oplus \tilde{H}_1(V) \to \tilde{H}_1(\Delta)$$

$$\to \tilde{H}_0(U \cap V) \to \tilde{H}_0(U) \oplus \tilde{H}_0(\Delta) \to 0.$$

□

On simplification, the sequence from Proposition 7.1 becomes:

$$0 \to 0 \to \tilde{H}_2(U) \oplus 0 \to \tilde{H}_2(\Delta)$$

$$\to 0 \to \tilde{H}_1(U) \oplus \tilde{H}_1(V) \to \tilde{H}_1(\Delta)$$

$$\to k^{342} \to \tilde{H}_0(U) \oplus \tilde{H}_0(V) \to \tilde{H}_0(\Delta) \to 0.$$

In particular:

Corollary 7.2 $\tilde{H}_2(\Delta) \cong \tilde{H}_2(U)$ *as vector spaces.*

□

U has all the 2-simplices of Δ. These are, as we have seen, of the form ${}^xB \leq {}^yA \leq {}^zS$, for some $x, y, z \in G$. The subgroup S contains exactly 3 conjugates of A and 3 conjugates of B. Each of these conjugates of A contains 2 conjugates of B, and each conjugate of B is contained in 14 conjuates of A, of which exactly 2 are contained in S. So each conjugate of S is a vertex of exactly $3 * 2 = 6$ 2-simplices, as we have seen. These 2-simplices adjoin each other to form a hexagon whose center is S and whose outer vertices are alternately conjugate either to A or B. Of course the complex is locally the same near every conjugate of S. We illustrate this in Figure 7.1. The labels on the vertices are not technically correct; where it says A,B, or S, it really shows some conjugate of that subgroup.

Now, we imagine "deforming" simultaneously all of these hexagons in U so that they all look like triangles, or barycentrically subdivided triangles, with a conjugate of S in the center of each triangle, a conjugate of A in the center of each edge, and a conjugate of B at each vertex, as shown in Figure 7.2.

Relabeling our picture, we get a homotopy-equivalent CW-complex with $3 * 7 = 21$ vertices (one for each conjugate of B), $3 * 7^2 = 147$ edges (one for each conjugate of

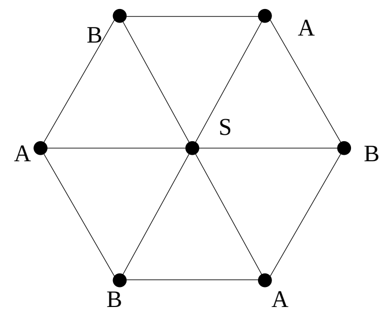

Figure 7.1: A local view of a subspace U of $|\mathcal{B}_3(W)|$, near a Sylow 3-subgroup S of W

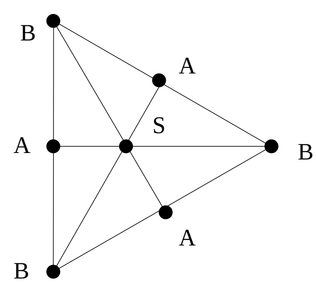

Figure 7.2: A local view of the subspace U of $|\mathcal{B}_3(W)|$, with edges deformed

A), and $7^3 = 343$ 2-cells (one for each conjugate of S), whose barycentric subdivision is our complex Δ. In this new complex E, two vertices are attached by an edge if and only if their corresponding subgroups (which are conjugate to B) are contained in the same conjugate of A. Each conjugate of B was contained in 14 conjugates of A, so in E each vertex is an endpoint for 14 edges; equivalently, each vertex of E neighbors 14 other vertices.

Proposition 7.3 *Let ${}^xB$ and ${}^yB$ be two distinct conjugates of B which do not neighbor B in E. Then ${}^xB$ and ${}^yB$ do not border each other in E.*

Proof: By computer. $\square$

Corollary 7.4 *The vertices of E are partitioned into three sets, each of which has no pair of vertices with an edge between them. That is, the vertices and edges of E form a* *tripartite graph. In fact, the vertices and edges of E form the complete tripartite graph $K_{7,7,7}$.*

Proof: Each of the 21 vertices of E has 14 neighbors, so each of the vertices of E has 6 other vertices which it does not neighbor, and which do not neighbor each other by Proposition 7.3. But this partite set cannot be any bigger than 7 vertices, since each vertex in this partite set neighbors all the remaining 14 vertices. $\square$

The graph $K_{7,7,7}$ has 7^3 3-cycles, and each 2-cell of E describes a 3-cycle in its vertices and edges, so E is completely described as having the vertices and edges of $K_{7,7,7}$ as well as a 2-cell for every 3-cycle of $K_{7,7,7}$. This complex is alternatively described as the join of a 7-vertex discrete space with itself three times. Thus we have proven:

Theorem 7.5 *U is homotopy equivalent to*

$$(\bigvee_{i=1}^{6} S^0) \star (\bigvee_{i=1}^{6} S^0) \star (\bigvee_{i=1}^{6} S^0).$$

$\square$

Corollary 7.6 *$\tilde{H}_2(\Delta; k) \cong k^{216}$.*

Proof:

$$\begin{aligned} U &\simeq E \\ &\simeq (\bigvee_{i=1}^{6} S^0) \star (\bigvee_{i=1}^{6} S^0) \star (\bigvee_{i=1}^{6} S^0) \\ &\simeq \bigvee_{i=1}^{6^3} S^2. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{H}_2(\Delta) &\cong \tilde{H}_2(U) \\ &\cong \tilde{H}_2(\vee_{i=1}^{216} S^2) \\ &\cong k^{216}. \end{aligned}$$

$\square$

Chapter 8

Explicit calculations of a Steinberg complex, part IV

As in the previous sections, let $W = G_{21} \wr C_3$, where G_{21} is up to isomorphism the unique non-abelian group of order 21, and let $p = 3$. We continue to find information about the Steinberg complex of W at $p = 3$ over the ring $R = \mathbb{F}_3$.

Proposition 8.1 *Let k be a field which is algebraically closed of characteristic 3. There are exactly eleven isomorphism classes of simple kG-modules: the trivial module, two of dimension 9, six of dimension 27, and two of dimension 81.*

Proof: By computer calculation, run in the program GAP (Groups, Algebras and Permutations):

```
gap> new;
Group([ (1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21),
(2,3,5)(4,7,6)(8,9,10,11,12,13,14)(15,21,20,19,18,17,16) ])
gap> Size(new);
27783
gap> tbl:=BrauerTable(new,3);
BrauerTable( Group([
(1,8,15)(2,9,16)(3,10,17)(4,11,18)(5,12,19)(6,13,20)(7,14,21),
(2,3,5)(4,7,6)(8,9,10,11,12,13,14)(15,21,20,19,18,17,16) ]), 3 )
gap> Display(tbl);
CT1mod3
```

```
         3  4  .  .  2  1  2  1  1  1  1  1
         7  3  3  3  3  3  3  3  3  3  3  3

           1a 7a 7b 7c 7d 7e 7f 7g 7h 7i 7j
        2P 1a 7a 7b 7c 7d 7e 7f 7g 7h 7i 7j
        3P 1a 7b 7a 7e 7f 7c 7d 7j 7i 7h 7g
        5P 1a 7b 7a 7e 7f 7c 7d 7j 7i 7h 7g
        7P 1a 1a 1a 1a 1a 1a 1a 1a 1a 1a 1a
       11P 1a 7a 7b 7c 7d 7e 7f 7g 7h 7i 7j
       13P 1a 7b 7a 7e 7f 7c 7d 7j 7i 7h 7g
       17P 1a 7b 7a 7e 7f 7c 7d 7j 7i 7h 7g
       19P 1a 7b 7a 7e 7f 7c 7d 7j 7i 7h 7g

X.1         1  1  1  1  1  1  1  1  1  1  1
X.2         9  A /A  E  2 /E  2  J  N /N /J
X.3         9 /A  A /E  2  E  2 /J /N  N  J
X.4        27 -1 -1  6  I  6 /I -1  6  6 -1
X.5        27 -1 -1  6 /I  6  I -1  6  6 -1
X.6        27  B /B  F -1 /F -1  K  L /L /K
X.7        27 /B  B /F -1  F -1 /K /L  L  K
X.8        27  C /C  G  6 /G  6  L  B /B /L
X.9        27 /C  C /G  6  G  6 /L /B  B  L
X.10       81  D /D  H -3 /H -3  M  O /O /M
X.11       81 /D  D /H -3  H -3 /M /O  O  M
```

□

Remark: The characters come in Galois-conjugate pairs. Over $\mathbb{F}_3$, which is not algebraically closed, it turns out that each nontrivial irreducible module is made up of one of these pairs of modules. Thus over $\mathbb{F}_3$, up to isomorphism there is one simple module of dimension 18, three of dimension 54, and one of dimension 162. Over the field of 9 elements, each of these modules decomposes into a direct sum of two absolutely irreducible modules.

Lemma 8.2 *W has a normal 3-complement of size* $7^3 = 343$.

Proof: $|W| = 3^4 * 7^3$, so a Sylow 7-subgroup of W has size $7^3 = 343$. By Sylow's Theorem, the number of Sylow 7-subgroups of W divides the W-index of a Sylow 7-subgroup, and is congruent to 1 mod 7. The index is $3^4 = 81$, and 1 is the only divisor of 81 which is congruent to 1 mod 7. Thus W has a unique Sylow 7-subgroup, which is therefore normal in W. $\square$

Corollary 8.3 *Each indecomposable projective kW-module P is isomorphic to the projective cover P_U of a unique (up to isomorphism) simple module U. For each simple U, P_U has composition factors U with multiplicity dim $P_U/dimU$. In particular, if U and T are simple kW-modules and $U \ncong T$, then $Hom(P_U, P_T) = 0$.*

Proof: This is part of Theorem 8.10 in [55]. $\square$

Proposition 8.4 *Let k be algebraically closed of characteristic 3. Then every indecomposable projective kW-module has k-dimension 81.*

Proof: We have the following identity (see [55], 8.3), where the sum is taken over a complete set of representatives of isomorphism classes of simple modules:

$$|W| = \sum_{U \text{ simple}} (\dim\ P_U * \dim\ U),$$

and $81 = |W|_3$ divides dim P_U for all U. If

$$\sum_U |W|_3 * \dim\ U = |W|,$$

then it follows that dim $P_U = |G|_3$ for all U. We have one simple U of dimension 1, two of dimension 9, six of dimension 27, and two of dimension 81. This gives us

$$81 * (1 + 2 * 9 + 6 * 27 + 2 * 81) = 81 * (343) = |W|.$$

So dim $P_U = 81$ for all U. $\square$

Corollary 8.5 *Let k be an algebraically closed field of characteristic 3, and let $S \in Syl_3(W)$. For every simple module kW-module U, the restricted module $(P_U) \downarrow_S \cong kS$.*

Proof: P_U is a projective kW-module, which implies (see [55], 8.2) that $(P_U) \downarrow_S$ is a projective kS-module. Every finitely-generated projective kS-module is free (see [55], 8.1), so a projective kS-module of k-dimension 81 is isomorphic to kS. $\square$

Definition 8.6 *Let R be a commutative ring with identity, G a finite group, $H \leq G$, and V an RG- module. We define the set function $Norm_H^R : V \to V$ by*

$$v \mapsto \sum_{g \in H} (g \cdot v).$$

Remark: We will often write Norm_H or Norm to mean Norm_H^R if there is no chance of confusion.

Remark: Norm is not generally an RG-module homomorphism: if $g \in G$, $v \in V$, and $H \leq G$, then $g \cdot \mathrm{Norm}_H(v) = \sum_{h \in H} (gh) \cdot v$, while $\mathrm{Norm}_H(g \cdot v) = \sum_{h \in H} (hg) \cdot v$. However:

Lemma 8.7 *Let R be a commutative ring with identity, H a finite group, and V an RH-module. Then $Norm_H : V \to V$ is an RH-module homomorphism.*

Remark: Lemma 8.7 applies in particular when V is an RG-module for some finite group G, and $H \leq G$.

The proof of Lemma 8.7 is a straightforward check of the module axioms. One starts by showing that $\mathrm{Norm}_H(V)$ is an R-module, and then checks that an arbitrary $h \in H$ maps $\mathrm{Norm}_H(V)$ into itself. The distributive and identity axioms are automatic because $\mathrm{Norm}_H(V) \subseteq V$. $\square$

Often we shall have V a permutation kG-module, so there is a basis B of V whose elements are permuted by the action of G on V, making B into a G-set.

Definition 8.8 *Let G be a finite group, let X be a G-set, and let $x \in X$. The* orbit *of x is the set $G \cdot x = \{g \cdot x | g \in G\}$. An orbit is* free *if it has the same cardinality as G.*

Lemma 8.9 *Let G be a finite group, let k be a field of characteristic $p > 0$, and let M be a finite-dimensional kG-permutation module, so that M is a k-vector space with basis $B = \{e_1, \ldots, e_n\}$ and B is a G-set. Then for any p-subgroup H of G, the linear transformation $Norm_H$ has rank equal to the number of free H-orbits of B.*

Example: Suppose $H = 1$. Then rank (Norm_H) = rank $(I_n) = n = |B|$, which is the number of H-orbits of B of size 1.

Proof of Lemma 8.9: First let $e_i \in B$, and assume $|H{*}e_i| < |H|$. Then $\mathrm{Stab}_H(e_i) \neq 1$, and since H is a p-group this implies that p divides $|\mathrm{Stab}_H(e_i)|$. So for every $h \in H$, the left coset $h\mathrm{Stab}_H(E_i)$ has size divisible by p. For all j, therefore, p divides $|\{g \in H | g * e_i = e_j\}|$. So for all j, the j^{th} coefficient of $\mathrm{Norm}_H(e_i)$ is divisible by p, and is therefore 0 when we take coefficients in a field k of characteristic p. So if $|H * e_i| < |H|$ then $\mathrm{Norm}_H(e_i) = 0$.

Now assume $|H * e_i| = |H|$, so the H-orbit of e_i is free. Then each element of H sends e_i to a different e_j, and therefore $\mathrm{Norm}_H(e_i) = \sum_{e_j \in H * e_i} e_j$. Reordering B appropriately, then, we see that as a matrix Norm_H will have blocks of 1's of size $|H| \times |H|$ along the diagonal, and one such block for each free H-orbit in B, and 0's elsewhere. $\square$

This proof also shows that Norm_H sends a permutation basis element to 0 if and only if its H-orbit is not free.

Definition 8.10 *Let Z be a ring with identity, and let U be a Z-module of finite composition length. If U is isomorphic to a direct sum of simple Z-modules, then we say that U is* semisimple.

Remark: With notation as in the above definition, it is a fact (see for instance [55], 1.8) that the sum

$$\sum_{V \text{ is a simple submodule of } U} V$$

of all simple submodules of U is semisimple. It is the unique largest semisimple submodule of U, and it is called the *socle* of U, written $\mathrm{Soc}(U)$. If $A \leq Z$, we often write $\mathrm{Soc}_A(U)$ to mean $\mathrm{Soc}((U) \downarrow_A)$.

Lemma 8.11 *Let k be a field of characteristic p and let G be a finite group with Sylow p-subgroup S. Suppose that P is an indecomposable projective kG-module such that the restricted module $(P) \downarrow_S$ is isomorphic to kS (as a kS-module). Then $Norm_S(P)$ generates $Soc_{kG}(P)$ as a kG-module.*

Remark: By ([55] 7.14), P is isomorphic to P_U, the projective cover of some simple kG-module U. Corollary 8.5 guarantees that the hypotheses of Lemma 8.11 are satisfied if $G = W$ and k is an algebraically closed field of characteristic 3.

Proof of Lemma 8.11: By assumption there exists an RS-module isomorphism $\alpha : (P) \downarrow_S \to kS$. Restricting α to $\mathrm{Norm}_S((P) \downarrow_S)$, we get a map $\bar{\alpha} : \mathrm{Norm}_S((P) \downarrow_S) \to$

kS, which sends $\sum_{g\in S}(g \cdot u) \mapsto \sum_{g\in S}(g \cdot \alpha(u))$. Thus $\bar{\alpha}$ maps $\mathrm{Norm}_S((P)\downarrow_S)$ into $\mathrm{Norm}_S(kS)$. We know that $\bar{\alpha}$ is injective, since it is a restriction of an injective map, and it is straightforward to check that the image of $\bar{\alpha}$ is exactly $\mathrm{Norm}_S(kS)$. Thus we have

$$\begin{aligned}\mathrm{Norm}_S(P) &= \mathrm{Norm}_S((P)\downarrow_S)\\ &\cong \mathrm{Norm}_S(kS),\end{aligned}$$

which is a kS-module by Lemma 8.7. By Lemma 8.9, $\mathrm{Norm}_S(kS)$ has dimension 1, and is therefore simple. So $\mathrm{Norm}_S(kS) \leq \mathrm{Soc}(kS)$.

It is a fact (see [55], 6.3 and 8.13) that $\mathrm{Soc}(kS)$ is simple, not just semisimple – in fact $\mathrm{Soc}(kS) \cong k$. Thus $\mathrm{Norm}_S(kS) = \mathrm{Soc}(kS)$. So we have:

$$\begin{aligned}\mathrm{Soc}(kS) &= \mathrm{Norm}_S(kS)\\ &\cong \mathrm{Norm}_S((P)\downarrow_S),\end{aligned}$$

so $\mathrm{Norm}_S((P)\downarrow_S)$ is a simple kS-module. Therefore it is a submodule of $\mathrm{Soc}((P)\downarrow_S)$, but since that module is isomorphic to the simple module $\mathrm{Soc}(kS)$ we have that $\mathrm{Norm}_S((P)\downarrow_S) = \mathrm{Soc}((P)\downarrow_S)$.

Finally, $\mathrm{Soc}_{kG}(P)$ is a nonzero kG-submodule of P; hence $(\mathrm{Soc}_{kG}(P))\downarrow_S$ is a nonzero kS-submodule of $(P)\downarrow_S$. Therefore there exists some simple kS-submodule Z of $(\mathrm{Soc}_{kG}(P))\downarrow_S$, which will then be a simple submodule of $(P)\downarrow_S$. Therefore $Z \leq \mathrm{Soc}((P)\downarrow_S)$, which implies that $Z = \mathrm{Soc}((P)\downarrow_S)$. Therefore $\mathrm{Soc}_{kG}(P) \supseteq \mathrm{Soc}((P)\downarrow_S) = \mathrm{Norm}_S((P)\downarrow_S) = \mathrm{Norm}_S(P)$. Since $\mathrm{Soc}_{kG}(P)$ is simple (see [55] 8.13), it is generated by $\mathrm{Norm}_S(P)$. $\square$

Proposition 8.12 *Let G be a finite group, let k be a field of characteristic $p > 0$, and let S be a Sylow p-subgroup of G. If P' is a kG-module with no non-zero projective summand, then $Norm_S(P') = 0$.*

Proof: We show the contrapositive. Suppose that $x \in P'$ and $\mathrm{Norm}_S(x) \neq 0$. There is a kG-homomorphism $f : kG \to P'$ such that $f(1) = x$. Now write $kG = P_1 \oplus \ldots \oplus P_d$ as a direct sum of indecomposable projective modules.

First suppose that for all i we have $\mathrm{Soc}(P_i) \subseteq \mathrm{Ker}(f)$. Then

$$\begin{aligned}
\mathrm{Norm}_S(x) &\in Norm_S(f(kG)) \\
&= f(\mathrm{Norm}_S(kG)) \\
&= f(\sum_{i=1}^{d} \mathrm{Norm}_S(P_i)) \\
&= f(\sum_{i=1}^{d} \mathrm{Soc}(P_i)) \\
&= 0,
\end{aligned}$$

a contradiction. Thus there is an i such that $\mathrm{Soc}(P_i) \cap \mathrm{Ker}(f) \neq \mathrm{Soc}(P_i)$. Since $\mathrm{Soc}(P_i)$ is simple, this means that f is injective on $\mathrm{Soc}(P_i)$.

It follows that f is injective on P_i: otherwise there would be a nonzero submodule of P_i which did not intersect $\mathrm{Soc}(P_i)$. Thus P' has a submodule isomorphic to P_i, which must then be a direct summand of P' since P_i is also injective: projective and injective modules are the same over kG. $\square$

Proposition 8.13 *Let k be a field of characteristic p, let $S \in Syl_p(G)$, and suppose that for all simple kG-modules U, dim $U \leq |S|$. If V is a semisimple kG-module, then $Norm_S(V)$ generates a maximal projective direct summand of V as a kG-module.*

Proof: Write $V = T_1 \oplus \ldots \oplus T_d$ where the T_i are simple kG-modules. Then $\mathrm{Norm}_S(V) = \bigoplus_i \mathrm{Norm}_S(T_i)$, and this generates

$$\bigoplus_{\mathrm{Norm}_S(T_i) \neq 0} T_i$$

as a kG-module by simplicity of the T_i. We claim that $\mathrm{Norm}_S(T_i) \neq 0$ if and only if T_i is projective as a kG-module: if T_i is projective then $T_i \downarrow_S$ is projective as a kS-module, and so T_i is a free kS-module and by dimensionality $T_i \downarrow_S \cong kS$. Therefore $\mathrm{Norm}_S(T_i) \neq 0$. Conversely, if $\mathrm{Norm}_S(T_i) \neq 0$ then by Proposition 8.12 $T_i \downarrow_S$ must have a projective direct summand since S is a p-group. Thus kS is a summand of T_i as a kS-module, and so by dimensionality $T_i \downarrow_S \cong kS$ and $T_i \downarrow_S$ is projective. Therefore T_i is projective as a kG-module. $\square$

Proposition 8.14 *Let $U = P \oplus P'$ be a direct sum of kW-modules, where P is projective and P' has no projective summand. Then $\mathrm{Norm}_S(U)$ generates the W-socle of P as a kW-module. This W-socle is uniquely determined, though P need not be.*

Proof: Write $P = Q_1 \oplus \ldots \oplus Q_n$ as a sum of indecomposable projectives. $\mathrm{Norm}_S(U) = \mathrm{Norm}_S(P) \oplus \mathrm{Norm}_S(P') = \mathrm{Norm}_S(P) \subseteq P$.

We show that $\mathrm{Norm}_S(Q_i)$ generates $\mathrm{Soc}(Q_i)$ as a kW-module, since then $\mathrm{Norm}_S(P)$ generates $\mathrm{Soc}(P)$.

As in Lemma 8.11, we have $\mathrm{Soc}_S(Q_i) \subseteq \mathrm{Soc}_W(Q_i)$. Then

$$\langle \mathrm{Norm}_S(Q_i) \rangle_S = \mathrm{Soc}_S(Q_i) \subseteq \mathrm{Soc}_W(Q_i),$$

and also $\mathrm{Soc}_W(Q_i)$, being simple, is generated by any nonzero vector it contains. $\square$

Recall that the Steinberg complex is the unique (up to isomorphism) minimal P_* that satisfies the conclusion of Theorem 3.1. Thus to find the Steinberg complex of W at $p = 3$, it is reasonable to start by isolating the projective summands of each permutation module in $\mathcal{B}_3(W)$. We can do this, according to the theory put forth in this section, by applying Norm_S to each permutation module in turn, and then taking the direct sum of the projective covers of the simple modules we get back.

We have done this in GAP, and we have calculated that the Steinberg complex should be chain-homotopic to the complex appearing below (where P_n is the projective cover of a simple module of dimension n, and the notation P_{n_i} is used to distinguish projective covers of nonisomorphic simple modules of dimension n when they exist). Note that P_n has dimension 162 in every case since we have taken coefficients in the field of three elements.

$$0 \to P_{54_1}^2 \oplus P_{162}^6 \to (P_{54_1} \oplus P_{162}^3) \oplus (P_{54_1} \oplus P_{162}^3) \oplus (P_{18} \oplus P_{54_1}^2 \oplus P_{54_2}^3 \oplus P_{54_3}^3 \oplus P_{162}^9)$$
$$\to P_{162}^2 \oplus P_{54_2} \to 0.$$

Since the Steinberg complex must be zero except in degrees which have nonzero homology above and below, we see that this complex is chain-homotopic to:

$$0 \to P_{54_1}^2 \oplus P_{162}^6 \to P_{18} \oplus P_{54_1}^4 \oplus P_{54_2}^2 \oplus P_{54_3}^3 \oplus P_{162}^{13} \to 0.$$

As vector spaces, this complex could be rewritten as:

$$0 \to (k^{162})^8 \to (k^{162})^{23} \to 0,$$

or

$$0 \to k^{1296} \to k^{3726} \to 0.$$

Segev and Webb have calculated (see [40]) that as vector spaces this complex Δ must have

$$\tilde{H}_2(\Delta) = k^{216}, \text{ and}$$
$$\tilde{H}_1(\Delta) = k^{2646}.$$

So the unique nonzero boundary map would have a kernel of dimension 216. We have calculated this kernel in GAP, and as a kW-module it is the direct sum of two indecomposable summands, one of dimension 162 and one of dimension 54. The boundary map is injective on the remaining summands, and so we have:

Theorem 8.15 *The Steinberg complex $St_3(W)$ is*

$$0 \to P_{54_1} \oplus P_{162} \to P_{18} \oplus P_{54_1}^3 \oplus P_{54_2}^2 \oplus P_{54_3}^3 \oplus P_{162}^8 \to 0,$$

where the complex is concentrated in degrees 1 and 2. The boundary map d_2 is zero on P_{162}, and it maps P_{54_1} into a summand isomorphic to P_{54_1}, with kernel the simple module S_{54_1}.

□

In particular, we see that indeed $\tilde{H}_2(\mathrm{St}_3(W))$ is not a projective kW-module (and $\tilde{H}_1(\mathrm{St}_3(W))$ is not projective either). This behavior of exhibiting non-projective homology is rare, as we will see in the next sections. It would be nice to have a better understanding of exactly when it does occur.

The endomorphism ring of this complex has morphisms in degrees -1, 0, and 1 which are not chain-homotopic to the zero map. For example, define a chain map $f : \mathrm{St}_3(W) \to \mathrm{St}_3(W)$ of degree 1 by sending everything to zero except a single P_{162}-summand in degree 1, which is sent by the identity to the P_{162}-summand in degree 2. This is a chain map, since d_2 restricted to P_{162} is the zero map, and d_1 is the zero map, so that $f_n \circ d_{n-1} = 0 = d_{n+1} \circ f_n$ for all n. But f is not chain-homotopic to the zero map: suppose $f \simeq 0$, so there exists $s : \mathrm{St}_3(W) \to \mathrm{St}_3(W)$ with $sd + ds = f$. Then s is of degree 2, but there are no nonzero morphisms of degree 2, a contradiction since $f \neq 0$.

Similarly, we may define $g \in \mathrm{End}_{kW}(\mathrm{St}_3)$ of degree -1 by setting g to be zero except on a single P_{162}-summand, on which g will be the identity. Then g is a chain map. Suppose $g \simeq 0$, so there exists $t \in \mathrm{End}_{kW}(\mathrm{St}_3)$ of degree 0 such that $td + dt = g$. Then since d_2 is zero on each P_{162}, on restricting g to the summand on which it is nonzero we have $\mathrm{Id}_{P_{162}} = g_2|_{P_{162}} = d_2t_2|_{P_{162}}$, so $d_2t_2 \neq 0$. This means that the image of $t_2|_{P_{162}}$ intersects P_{54_1} nontrivially, which says that $\mathrm{Hom}_{kW}(P_{162}, P_{54}) \neq 0$, a contradiction. So we have:

Theorem 8.16 *End($St_3(W)$) in the homotopy category of complexes of kW-modules is nonzero in dimensions -1, 0, and 1.*

We remark also that these maps do not induce homology isomorphisms: one can either see this directly, or apply Theorem 3.12. Therefore we have established:

Corollary 8.17 *The Steinberg complex is not in general a (partial) tilting complex:* i.e.*, a complex P_* of finitely-generated projective modules such that Hom$(P_*, P_*[i]) = 0$ for all $i \neq 0$ in the derived category.*

Corollary 8.18 *The Steinberg complex construction is not a functor from the category of finite groups to the category of R-modules.*

Proof: Let G be a finite group and p a prime such that $\mathrm{St}(G; \mathbb{F}_p) \neq 0$ (for example, take $G = W$ and $p = 3$ as above). Then we have an inclusion map of finite groups $G \to G \times C_p$, and a quotient map $G \times C_p \to G$, whose composition is the identity on G. However, $\mathrm{St}(G \times C_p; \mathbb{F}_p) = 0$ since $O_p(G \times C_p) = C_p$. Thus if $\mathrm{St}(-; \mathbb{F}_p)$ were a functor (covariant or contravariant) from the category of finite groups to $\mathbb{F}_p$-mod, then $\mathrm{Id}_{\mathrm{St}(G;\mathbb{F}_p)}$ would factor through the zero complex. $\square$

On the other hand, $\mathrm{St}(-)$ is a functor on the category with objects all finite groups and morphisms all injections. The proof is, roughly, that each step of the construction is then functorial: from (Groups) to (posets with a group action) to (simplicial complexes with a group action) to (chain complexes). The trouble with allowing non-injective maps is that a morphism of groups can send a nonidentity p-subgroup to the identity subgroup, which we do not accept as part of our poset of p-subgroups.

We will see in section 10 that there are sometimes other ways to "build up" subgroup complexes from those of smaller groups.

Chapter 9

A subgroup of this wreath product

We choose to focus on the non-projective part of $\tilde{H}_2(\mathrm{St}_3(W))$, the S_{54_1} term. Let M denote this module. We look for a subgroup H of W, such that $M\downarrow_H\uparrow^W$ has a summand isomorphic to M, written $M|(M\downarrow_H\uparrow^W)$.

Definition 9.1 *Let R be a commutative ring with identity, let G be a finite group, and let H be a subgroup of G. An RG-module M is said to be H-*projective*, or* projective relative to *H, if there exists an RH-module V such that $H|(V\uparrow_H^G)$.*

Example: If R is a field, then every 1-projective RG-module is projective.

So we are looking in particular for a subgroup H such that M is H-projective. Such a subgroup is $H = (C_7 \times C_7 \times C_7) \rtimes (\delta(C_3) \times C_3)$, where $\delta(C_3)$ is the diagonal subgroup of order 3 in the base group of W, so $|G : H| = 9$.

A GAP calculation of the Brauer character table of H in characteristic 3 yields that over a splitting field k of characteristic 3, there are exactly 36 simple kH-modules of dimension 9, 6 of dimension 3, and 1 of dimension 1. Over $\mathbb{F}_3$, as with W, the nontrivial simples are made up of conjugate pairs of these modules, so that there are 3 of dimension 6 and 18 of dimension 18. Much of the same theory that we used in describing W applies also to H; in particular, the projective cover of each of these modules has dimension 18.

Let $T = \delta(C_3) \times C_3$, and suppose that $1 \lneq A \lneq T$ and ${}^xA \leq {}^yT$, so that $xAx^{-1} \leq yTy^{-1}$. Then $y^{-1}xAx^{-1}y \leq T$, which implies that $y^{-1}xAx^{-1}y = A$, since T contains only one subgroup which is H-conjugate (in fact, W-conjugate) to A, namely A itself. It follows that ${}^yA = yy^{-1}xAx^{-1}yy^{-1} = {}^xA$. So there exists a unique H-orbit of simplices ${}^xA \leq {}^yT$ for each such A. Finally there are exactly 3 such A satisfying $O_3(N_H(A)) = A$, and so we calculate that the Steinberg complex of H over $\mathbb{F}_3$ is chain-homotopic to:

$$0 \to \oplus_{j=1}^{3}(\oplus_{i=1}^{18} P_{18_i}) \to (\oplus_{i=1}^{18} P_{18_i}) \oplus (P_{18_1} \oplus P_{18_2}) \oplus (P_{18_3} \oplus P_{18_4}) \oplus (P_{18_5} \oplus P_{18_6}) \to 0,$$

for a suitable numbering of the P_{18}'s. As k-vector spaces, where $k = \mathbb{F}_3$, this is

$$0 \to k^{972} \to k^{432} \to 0.$$

Let d denote the boundary map between the nonzero chain groups C_1 and C_0. Each P_{18_i} appears twice in $\mathrm{Ker}(d)$, except those which appear twice in C_0. So d is onto. (Alternatively, we could have said that the dimension of the kernel is $540 = 972 - 432$.) Each P_{18_i} is simple, so $\mathrm{St}_3(H) =$

$$0 \to (\oplus_{i=1}^{18} P_{18_i}) \oplus (\oplus_{i=7}^{18} P_{18_i}) \to 0.$$

In particular, this complex does not exhibit non-projective homology, even though the non-projective homology of $\mathrm{St}_3(W)$ behaves well with respect to this subgroup.

Chapter 10

Another Steinberg complex with non-projective homology

This section will follow a paper of Segev and Webb [40]. We show that $\tilde{H}_3(\mathcal{A}_2(G))$ is not projective, where $G = (S_3 \times S_3 \times S_3 \times S_3) \rtimes C_2$, where C_2 acts on the base group $N = S_3 \times S_3 \times S_3 \times S_3$ by permuting the first two factors and the last two factors. (Thus G is isomorphic to a subgroup of $S_3 \wr C_4$ of index 2. $S_3 \wr C_4$ is a group whose Steinberg complex exhibits non-projective homology – see [44]. We thank Prof. Jon Carlson for suggesting this example.)

Let $\mathcal{A}_p(G)$ denote the G-poset of nonidentity elementary abelian p-subgroups of G, and let $\mathcal{A}_p(G)_N$ denote the poset obtained by adding to $\mathcal{A}_p(G)$ an additional element 0, such that $0 < A$ for all $A \in \mathcal{A}_p(G)$ such that $A \cap N \neq 1$. We will not actually use $\mathcal{A}_p(G)_N$ here, except to state and apply the following theorem.

Theorem 10.1 *(Main Theorem of [40].) Let G be a finite group and p a prime. Suppose that N is a normal subgroup of G such that p divides $|N|$. Further except in (1) assume that if A is an elementary abelian p-subgroup of G with $A \cap N = 1$ then A is cyclic, and let $\mathcal{M} = \{A \in \mathcal{A}_p(G) | A \cap N = 1\}$.*

(1) There exists a long exact sequence of $\mathbb{Z}G$-modules

$$\ldots \to \tilde{H}_n(\mathcal{A}_p(N)) \to \tilde{H}_n(\mathcal{A}_p(G)) \to \tilde{H}_n(\mathcal{A}_p(G)_N) \to \tilde{H}_{n-1}(\mathcal{A}_p(N)) \to \ldots$$

(3) For $n \geq 0$,

$$\tilde{H}_n(\mathcal{A}_p(G)_N) \cong \bigoplus_{A \in \mathcal{M},\ \textit{up to conjugacy}} \tilde{H}_{n-1}(\mathcal{A}_p(C_N(A))) \uparrow_{N_G(A)}^G$$

as $\mathbb{Z}G$-modules.

(We omit mention of conclusions (2) and (4) of their theorem here.)

We set $p = 2$ and let N and G be the groups described above, and we set $V = \tilde{H}_0(\mathcal{A}_2(S_3))$, the 0-dimensional reduced homology (with integer coefficients) of the poset of nonidentity elementary abelian 2-subgroups of the symmetric group on 3 letters. $\mathcal{A}_2(K)$ is a K-poset for every group K (with the K-action given by conjugation of subgroups), so V is a $\mathbb{Z}S_3$-module. Thus it may be considered a $\mathbb{Z}N$-module by projecting N onto its first factor, and may also be considered a $\mathbb{Z}[\Delta(S_3) \times C_2]$-module by $(\Delta(a), b) \cdot v = a \cdot v$.

We want to apply the Main Theorem of [40] to G, so we first check that its hypotheses hold. The hypotheses for (1) are clearly true. For (3), we assume that A is an elementary abelian 2-subgroup of G such that $A \cap N = 1$, and show that A is cyclic. Let $\alpha \in A$, and let $\pi_{C_2} : G \to G/N = C_2 = \langle x \rangle$ be the canonical projection.

If $\pi_{C_2}(\alpha) = 1$ then $\alpha \in N \cap A = 1$ and $\alpha = 1$, so let $\alpha, \beta \in A - \{1\}$ (of course we may assume $A \neq 1$). Then $\pi_{C_2}(\alpha) = \pi_{C_2}(\beta) = x$, so $\pi_{C_2}(\alpha\beta) = 1$. Therefore by the above argument, $\alpha\beta = 1$ and $\beta = \alpha^{-1} \in \langle \alpha \rangle$ (in fact $\beta = \alpha$). Thus $A = \langle \alpha \rangle$ is cyclic, so part (3) of the Main Theorem also holds.

By (1) we have a long exact sequence of $\mathbb{Z}G$-modules:

$$\ldots \to \tilde{H}_n(\mathcal{A}_2(N)) \to \tilde{H}_n(\mathcal{A}_2(G)) \to \tilde{H}_n(\mathcal{A}_2(G)_N) \to \tilde{H}_{n-1}(\mathcal{A}_2(N)) \to \ldots.$$

Next, by (3) we have that for all $n \geq 0$,

$$\tilde{H}_n(\mathcal{A}_2(G)_N) \cong \bigoplus_{A \in \mathcal{M},\ \text{up to conjugacy}} \tilde{H}_{n-1}(\mathcal{A}_2(C_N(A))) \uparrow_{N_G(A)}^G$$

as $\mathbb{Z}G$-modules.

We calculate $\mathcal{M}$: Suppose $A \in \mathcal{M}$. Then $A = \langle \alpha \rangle \cong C_2$, and $\pi_{C_2}(\alpha) = x$. Thus $\alpha = (a, a^{-1}, b, b^{-1}, x)$ for some $a, b \in S_3$, and this is conjugate to $(1, 1, 1, 1, x)$ by $(a, 1, b, 1, 1)$ under some convention. So up to conjugacy, $\mathcal{M}$ consists of one subgroup of G, namely $\langle x \rangle$. We let $A = \langle x \rangle$.

Next we observe that $C_N(A) = \Delta_{12}(S_3) \times \Delta_{34}(S_3)$, where $\Delta_{ij}(S_3)$ is the subgroup of N consisting of elements whose ith and jth entries are identical, and whose other entries are 1. Finally $N_G(A) = C_G(A) = \Delta_{13}(S_3) \times \Delta_{24}(S_3) \times \langle x \rangle$.

Thus at $n = 3$ the long exact sequence above becomes:

$$\tilde{H}_3(\mathcal{A}_2(S_3 \times S_3)) \uparrow_{S_3 \times S_3 \times A}^{G} \to \tilde{H}_3(\mathcal{A}_2(N))$$

$$\to \tilde{H}_3(\mathcal{A}_2(G)) \to \tilde{H}_2(\mathcal{A}_2(S_3 \times S_3)) \uparrow_{S_3 \times S_3 \times A}^{G}$$

But we note that $\tilde{H}_n(\mathcal{A}_2(S_3 \times S_3)) = 0$ for $n \geq 2$. This is because $\mathcal{A}_2(S_3 \times S_3) \simeq \mathcal{A}_2(S_3) \star \mathcal{A}_2(S_3)$, where $\star$ denotes the topological join, and $\mathcal{A}_2(S_3)$ is a discrete set of 3 points, so that its join with itself is a graph. Therefore $\tilde{H}_3(\mathcal{A}_2(N)) \cong \tilde{H}_3(\mathcal{A}_2(G))$. But now we can argue as in Proposition 5.2 of [40] that $\mathcal{A}_2(N)$ is the join of 4 copies of $\mathcal{A}_2(S_3)$, which is a wedge of $2^4 = 16$ spheres of dimension $4 - 1 = 3$.

Then $\tilde{H}_0(\mathcal{A}_2(G))$ has a basis given by $\{a - b, b - c\}$, where a, b, c are the three nonidentity 2-subgroups of S_3. And $\tilde{H}_3(\mathcal{A}_2(G))$ has a basis, given by $\{v_1 \otimes v_2 \otimes v_3 \otimes v_4 | v_i \in \{a - b, b - c\}$.

Finally we let k be a field of characteristic 2 and claim that $\tilde{H}_3(\mathcal{A}_2(G)) \otimes k$ is not a projective kG-module. If it were, then its restriction to any subgroup would preserve projectivity. However, the basis we gave above has an element which is fixed by A: namely, the element $(a - b) \otimes (a - b) \otimes (a - b) \otimes (a - b)$, which says that $\tilde{H}_3(\mathcal{A}_2(G)) \downarrow_A$ has a trivial summand. This is not projective since projective kC_2-modules must have k-dimension divisible by 2.

Chapter 11

A minimal group with non-projective Steinberg complex homology

In the last section, we examined a group $B = (S_3 \times S_3 \times S_3 \times S_3) \rtimes C_2$, where $A = C_2$ acts on the base group $N = S_3^4$ by permuting the first two factors and the last two factors. We found that if k is a field of characteristic 2, then $\tilde{H}_3(\mathcal{A}_2(B); k)$ is not projective. Equivalently, the Steinberg complex of B at $p = 2$ has nonprojective homology in degree 3.

This nonprojectivity condition seems to be somewhat rare. In this section, we will show that if p and q are prime numbers and J is a group such that $|J| = p^a q^b < |B|$, then the Steinberg complex of J at any prime will have projective homology in all dimensions. Note that $|B| = 2^5 * 3^4 = 2592$.

Suppose G is a finite group of order < 2592 and p is a prime. Unless noted, by $\mathrm{St}_p(G)$ we will mean the Steinberg complex $\mathrm{St}_*(G; \mathbb{F}_p)$, and k will denote $\mathbb{F}_p$. Of course if $p \nmid |G|$ then $\mathrm{St}_p(G)$ is the trivial kG-module k concentrated in degree -1, and this module is projective since $p \nmid |G|$. On the other extreme, if G is a p-group, or more generally if $O_p(G) \neq 1$, then the p-subgroups complex $\mathcal{S}_p(G)$ is contractible, so the Steinberg complex of G at p is the 0 complex.

Lemma 11.1 *Suppose G is a finite group such that $p^3 \nmid |G|$. Then $H_n(St_3(G))$ is* *projective for all n.*

Proof: $\mathrm{St}_p(G)$ is a complex $\cdots \to P_2 \to P_1 \to P_0 \to 0$ of projective $\mathbb{F}_pG$-modules. Let $d_n : P_n \to P_{n-1}$ be the differential maps of this complex, and let $X = \mathrm{St}_p(G)$, so $H_n(X) \cong \tilde{H}_n(\mathcal{S}_p(G))$ for all n. We know [51] that $P_0/\mathrm{Im}(d_1) = H_0(X)$ is projective, so we have a short exact sequence

$$0 \to \mathrm{Im}(d_1) \to P_0 \to P_0/\mathrm{Im}(d_1) \to 0,$$

whose fourth term is projective. Thus $P_0 = \mathrm{Im}(d_1) \oplus H_0(X)$, so $\mathrm{Im}(d_1)$ is projective, and so $X = (\cdots \to P_2 \to P_1 \to \mathrm{Im}(d_1) \to 0) \oplus (0 \to H_0(X) \to 0)$. If $p^3 \nmid |G|$ then $\mathcal{S}_p(G)$ is a graph, so $P_2 = 0$. Then we have a short exact sequence

$$0 \to \mathrm{Ker}(d_1) \to P_1 \to \mathrm{Im}(d_1) \to 0,$$

which splits since $\mathrm{Im}(d_1)$ is projective. But $\mathrm{Ker}(d_1) = H_1(X)$ since $\mathrm{Im}(d_2) = 0$, so $H_1(X)$ is a summand of P_1 and therefore projective. □

Lemma 11.2 *Suppose G is a finite group with no elementary abelian p-subgroup of rank 3. Then $H_n(St_p(G))$ is projective for all n.*

Proof: We could just as well have used $\mathcal{A}_p(G)$ in place of $\mathcal{S}_p(G)$ in the proof above. □

So in searching for groups G such that $\mathrm{St}_p(G)$ has non-projective homology, we may assume that G has an elementary abelian p-subgroup of order p^3.

The next lemma is very similar, so we include it here:

Lemma 11.3 *Suppose that $\tilde{H}_n(\mathcal{S}_p(G)) = 0$ for all $n > M$ and for all $n < m$. Then $St_p(G)$ is concentrated in degrees between m and M (inclusive).*

Proof: $\mathcal{S}_p(G)$ is bounded above and below, so $\mathrm{St}_p(G)$ is also. Then use the fact that the module P_i in degree i of $\mathrm{St}_p(G)$ is both projective and injective to split off contractible summands from both ends as long as the complex is exact at that degree. □

After the case of p not dividing the order of G and the case of G being a p-group, the next-simplest case is to suppose that $|G| = p^a q^b$ for $a, b \geq 1$ and p, q distinct primes. The following theorem is stated twice; the only purpose for this is to avoid confusing the author (who is treating p as fixed and therefore has trouble switching p and q).

Theorem 11.4 *("Burnside's* other p^aq^b *theorem" [19].) Suppose G is a finite group of order p^aq^b where p and q are distinct primes.*

*(A.) If $p^a > q^b$ then at least one of the following is true: (A0) $O_p(G) \neq 1$, or (A1) $p = 2$ and q is a Fermat prime (*i.e.*, $q = 3, 5, 17$, or 257), or (A2) p is a Mersenne prime (*i.e.*, $p = 3, 7, 31$, or 257, or $p \geq 8191$) and $q = 2$.*

(B.) If $q^b > p^a$ then at least one of the following is true:

*(B0) $O_q(G) \neq 1$, or (B1) $q = 2$ and p is a Fermat prime (*i.e.*, $p = 3, 5, 17$, or 257), or (B2) q is a Mersenne prime (*i.e.*, $q = 3, 7, 31$, or 257, or $q \geq 8191$), and $p = 2$.*

Of course if (A0) is true then $\mathrm{St}_p(G) = 0$.

We also have Glauberman's analogous theorem [19], which is harder to apply but applies to all cases:

Theorem 11.5 *Let G be a group of order p^aq^b as above, and let P be a Sylow p-subgroup of G and Q a Sylow q-subgroup of G. For any subgroup $A \leq G$, let $e(A)$ be the maximum of the orders of the subgroups of A having nilpotence class 2 or less. If $e(P) > e(Q)$ then $O_p(G) \neq 1$. (And if $e(Q) > e(P)$ then $O_q(G) \neq 1$.)*

We thus assume G is a group of order $p^aq^b < 2592$ such that $\mathrm{St}_p(G)$ exhibits non-projective homology. By the results above we may assume that $a \geq 3$ and either:

i) $q^b > p^a$, or ii) $p^a > q^b$ and (A1) holds, or iii) $p^a > q^b$ and (A2) holds.

First suppose case (iii) holds. Obviously $p^3 < 2592$ so $p = 3$ or 7. If $p = 7$, then we can assume moreover that G has order $7^3 * 2^b$ for $b \leq 2$, since $7^4 = 2401 > 2592/2$, and $2^3 * 7^3 = 2744 > 2592$. If $p = 3$, then our $3^a * 2^b$ must divide either $3^6 * 2 = 1458$, $3^5 * 2^3$, or $3^4 * 2^5$, and since $3^4 * 2^5 = 2592$ is our "index case" we may assume G has size strictly less than $3^4 * 2^5$. All of the cases above except $3^3 * 2^5$ are eliminated by Glauberman's $e(P) > e(Q)$ condition, since all groups of order p^3 have nilpotence class 2 or less, so we are left to consider a group of order $3^3 * 2^5$ whose Sylow 2-subgroup has nilpotence class 2 or less. Further, such a group must have an elementary abelian Sylow 3-subgroup.

In case (ii), we have that $p^a > q^b$ and $p = 2$ and $q = 3, 5, 17$, or 257. Thus $q^b < \sqrt{2592} < 51$, so q^b is either $17, 5^2$ or smaller, or 3^3 or smaller. The choices here are then $2^a * 17$ for $a = 5, 6, 7$, or $2^a * 5$ for $a = 3, 4, \ldots, 9$, or $2^a * 5^2$ for $a = 5, 6$, or $2^a * 3^3$ for $a = 5, 6$, or $2^a * 3^2$ for $a = 4, \ldots 8$, or $2^a * 3$ for $a = 3, \ldots, 9$.

Glauberman's condition eliminates $2^i * 5$ and $2^i * 3$ for all i, as above. Also, if the order of G is $2^i * 17$, then $n_{17} = 1$ (mod) 17, and n_{17} divides 2^7, so $n_{17} = 1$ and in this case the q-Sylow subgroup Q is normal in G and the rank of $Q/\Phi(Q)$ (where $\Phi(Q)$ is the Frattini subgroup of Q) is less than the rank of a p-Sylow subgroup, and this case is eliminated below. So the choices remaining are $2^a * 5^b$ for $a = 5, 6, b = 2, 3$, or $2^a * 3^2$ for $a = 4, \ldots, 8$. Of these, the GAP function `AllSmallGroups` would in principle apply to all sizes except $2^8 * 3^2$.

A GAP calculation shows that all groups of order 32 have a subgroup of order ≥ 16 and nilpotence class ≤ 2, and so the cases $2^5 * 3^2, ..., 2^8 * 3^2$ are all eliminated (since a group of order p^i has a subgroup of size $2^5 = 32$ for all $i \geq 5$, and in turn this subgroup has a subgroup of order 16 of nilpotence class 1 or 2).

In the same way we also check the cases $2^6 * 5^2$ and $2^6 * 3^3$ by checking that every group of order 2^6 has a subgroup of size $2^5 = 32 > 27 = 3^3$ of nilpotence class < 3. (It took GAP less than two minutes to enumerate all 267 groups of order 64, and then to run a simple and doubtless inefficient program that checked this subgroup property.) This leaves $2^5 * 5^2 = 800, 2^5 * 3^3 = 864$, and $2^4 * 3^2 = 144$ as potential sources of difficulty (where in each case the Sylow 2-subgroup must have nilpotence class 3 or 4). We remark, however, that so far we have tested only for acyclicity, which is certainly stronger than the condition of exhibiting projective homology. So if these "nearly-contractible" cases exhibited non-projective homology, I think it would be somewhat surprising.

A quick GAP search eliminates 144 from consideration:

```
gap> all:=AllSmallGroups(144);;
gap> temp:=[];;
gap> for i in all do
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;
> od;
gap> Size(temp);
1
gap> g:=temp[1];
<pc group of size 144 with 6 generators>
gap> b2:=[];;
gap> reps:=ConjugacyClassesSubgroups(g);;
gap> reps:=List(reps,Representative);;
```

```
gap> for i in reps do
> if PCore(Normalizer(g,i),2) = i then
> if Size(i) > 1 then
> Add(b2,i);
> fi;
> fi;
> od;
gap> b2;
[ Group([ f2 ]), Group([ f1, f2, f3, f4 ]) ]
```

So there is at most (or exactly, assuming Quillen's Conjecture holds) one group of order 144 which exhibits homology, but its 2-subgroups complex is G-homotopy equivalent to a graph so its homology is projective.

We next consider groups of order 800:

```
gap> all:=AllSmallGroups(800);;
gap> temp:=[];;
gap> for i in all do
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;
> od;
gap> Size(temp);
1
gap> g:=temp[1];
<pc group of size 800 with 7 generators>
gap> reps:=ConjugacyClassesSubgroups(g);;
gap> reps:=List(reps,Representative);;
gap> s2:=[];;
gap> for i in reps do
> if Size(i) in DivisorsInt(32) then
> if Size(i) > 1 then
> Add(s2,i);
> fi;
> fi;
> od;
gap> Collected(List(s2,Size));
[ [ 2, 3 ], [ 4, 7 ], [ 8, 7 ], [ 16, 3 ], [ 32, 1 ] ]
gap> a2:=[];;
gap> for i in s2 do
> if IsElementaryAbelian(i) then
```

```
> Add(a2,i);fi;od;
gap> Collected(List(a2,Size));
[ [ 2, 3 ], [ 4, 2 ] ]
```

So there is one group of order 800 whose 2-subgroups complex has homology, and by the lemma above concerning elementary abelian p-subgroups, the homology of that complex is projective.

Finally we consider $|G| = 864$: We go through the same GAP calculations, and again there is exactly one group of this order which has homology. Again we consider its elementary abelian subgroups and its 2-radical subgroups. This time neither of these complexes is a graph.

```
gap> all:=AllSmallGroups(864);;
gap> temp:=[];;
gap> for i in all do
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;
> od;
gap> Size(temp);
1
gap> g:=temp[1];
<pc group of size 864 with 8 generators>
gap> reps:=ConjugacyClassesSubgroups(g);;
gap> reps:=List(reps,Representative);;
gap> s2:=[];;
gap> for i in reps do
> if Size(i) in DivisorsInt(32) then
> if Size(i) > 1 then
> Add(s2,i);fi;fi;od;
gap> Collected(List(s2,Size));
[ [ 2, 5 ], [ 4, 9 ], [ 8, 11 ], [ 16, 7 ], [ 32, 1 ] ]
gap> a2:=[];;
gap> for i in s2 do
> if IsElementaryAbelian(i) then
> Add(a2,i);
> fi;od;
gap> Collected(List(a2,Size));
[ [ 2, 5 ], [ 4, 5 ], [ 8, 1 ] ]
gap> b2:=[];;
```

```
gap> for i in s2 do
> if PCore(Normalizer(g,i),2)=i then
> Add(b2,i);fi;od;
gap> Collected(List(b2,Size));
[ [ 2, 2 ], [ 4, 1 ], [ 16, 1 ], [ 32, 1 ] ]
```

However, we can exhibit a group of order 864 with trivial 2-core: let $E = C_3 \times C_3$, and let S be a Sylow 2-subgroup of $\mathrm{Aut}(E) = GL_2(\mathbb{F}_3)$, which has order $2^4 = 16$ since $GL_2(\mathbb{F}_3)$ has order $(3^2-1)*(3^2-3) = 8*6 = 48 = 2^4*3$. Thus $E \rtimes S$ has order $3^2 * 2^4$. Finally let $G = (E \rtimes S) \times S_3$, so that $|G| = 2^5 * 3^3 = 864$. Then $O_2(G) = O_2(E \rtimes S) = \cap_{Q \in \mathrm{Syl}_2(E \rtimes S)} Q \leq S$. But every nonidentity element $f \in S$ moves some vector $v \in E$, so $f(v) - v \neq 0$ (and so $(v,1)*(0,f)*(v,1)^{-1}$ has a nonzero E-component). Thus $O_2(G) = 1$. GAP has told us that this is the only such group of order 864. It remains to show that the Steinberg complex of this group has projective homology.

We must prove a proposition:

Proposition 11.6 *If p, q are distinct primes, then $GL_n(q)$ has elementary abelian p-subgroups of size at most p^n.*

Proof: Let $P = C_p^r$ be an elementary abelian p-subgroup of $GL_n(q)$. P acts on $C_q^n = \mathbb{F}_q^n$ completely reducibly since q does not divide the order of P. Thus $\mathbb{F}_q^n = S_1 \oplus \ldots \oplus S_k$ as $\mathbb{F}_q P$-modules, where each S_i is simple. But any irreducible representation of P is a repesentation for a cyclic image of P, by Schur's Lemma and the existence of maps $C_p^r \to C_p \to C_p^r$. Thus $k \geq r$, but each S_k has dimension ≥ 1 so $k \leq n$. Thus $n \geq r$. $\square$

This says in particular, with E, S as above, that $\mathcal{A}_2(S)$ is a graph, and therefore $\mathcal{A}_2(E \rtimes S)$ is a graph. By [33] it is a disconnected graph if and only if $E \rtimes S$ has a strongly-embedded 2-subgroup: that is, a subgroup M such that $S \leq M < E \rtimes S$, and such that $M^x \cap M$ has odd order for all $x \notin M$. But S fixes no proper subspace of E, so such an M must be equal to S. However, if we let $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then ${}^xS \cap S =$ $\{s(x) - x, s) | s \in S, s(x) = x\} = \{(0,s) | s \in S, s\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \geq \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cong C_2$. Thus $S \cap {}^xS$ has even order.

Therefore $\mathcal{A}_2(E \rtimes S)$ has reduced homology in at most one degree. We also know that $\mathcal{A}_2(S_3)$ is discrete. Thus $\mathcal{A}_2((E \rtimes S) \times S_3) \simeq \mathcal{A}_2(E \rtimes S) \star \mathcal{A}_2(S_3)$ is a wedge of copies of the suspension of $\mathcal{A}_2(E \rtimes S)$, which has reduced homology only in one degree. Therefore the Steinberg complex has projective homology, since the Steinberg complex is concentrated between the maximum and minimum degrees which exhibit nonzero homology. (It is worth pointing out here that a d-dimensional simplicial complex has torsion-free integral homology in dimension d, since the dth homology group is just the kernel of the dth boundary map, and any subgroup of a free abelian group is free abelian. Thus by the Universal Coefficients Theorem, $\tilde{H}_n(\mathcal{A}_2((E \rtimes S) \times S_3); k) \cong k \otimes \tilde{H}_n(\mathcal{A}_2((E \rtimes S) \times S_3); \mathbb{Z})$ for all n.)

The remaining possibility is (i), so $q^b > p^a$. Thus $p^3 \leq p^a < \sqrt{2592} = 72/\sqrt{2} < 51$, so $p \leq 3$. Moreover, if $p = 3$ then $a = 3$. Now $q^b < 2592/27 = 96$, but $q^b > p^a = 27$. So $27 < q^b < 96$. Now, if $q \neq 2$ and $q \neq p = 3$, then $q \geq 5$. But then $q^3 > 96$, so we have $b \leq 2$.

Suppose that $b = 1$. Then $q > 27$, so by Sylow's Theorem, $n_q \equiv 1 \bmod q$ and $n_q | 27$. Thus $n_q = 1$ and Q is normal in G. (We could also have said that $O_q(G) \neq 1$ by "Burnside's other theorem" stated above, and G has q-rank 1 so $O_q(G) = Q$ is a Sylow q-subgroup of G.) On the other hand, if $b = 2$ then $27 < q^b < 96$ implies that $q = 7$, and in this case $n_q | 27$ and $n_q \equiv 1 \bmod 7$ also implies $n_q = 1$. (Here the relevant condition is that $a < q - 1$: p is a generator for the multiplicative group $\mathbb{F}_q^\times \cong C_{q-1}$. $n_q = 1 = p^0$ satisfies both conditions of Sylow's Theorem. But the divisors of p^a are $1, p, \ldots, p^a$, and if $a < q - 1$ then these will all have different residues mod q, so $n_q = 1$ is the only possibility.) So in either case $n_q(G) = 1$, and $G = Q \rtimes P$ where $Q \in \mathrm{Syl}_q(G), P \in \mathrm{Syl}_p(G)$.

Now, if $G = Q \rtimes P$ where Q and P are Sylow q- and p-subgroups of G, respectively, then there is a group homomorphism $\phi : P \to \mathrm{Aut}(Q)$. Moreover, $\mathrm{Ker}(\phi)$ is a normal subgroup of P which also normalizes Q (because the action of P on Q is conjugation in G). Thus $\mathrm{Ker}(\phi)$ is a normal p-subgroup of G. We therefore assume it is the identity, so ϕ is injective.

For any characteristic subgroup Y of a group J there is a group homomorphism $\pi : \mathrm{Aut}(J) \to \mathrm{Aut}(J/Y)$, given by $\pi(f)(xY) = f(x)Y$. Moreover, in the case that $J = Q$ is a q-group and $Y = \Phi(Q)$ is the Frattini subgroup of Q, it is well-known (see [20], Theorem 5.1.4) that $\mathrm{Ker}(\pi)$ is a q-group. Thus $\mathrm{Ker}(\pi) \cap \mathrm{Im}(\phi) = 1$, so $\mathrm{Ker}(\pi \circ \phi) = \{x \in P | \phi(x) \in \mathrm{Ker}\pi\} = \{x \in P | \phi(x) = 1\} = \mathrm{Ker}(\phi) = 1$.

Thus P is isomorphic to a subgroup of $\mathrm{Aut}(Q/\Phi(Q))$. But $Q/\Phi(Q)$ is elementary abelian of rank n for some n, so P is isomorphic to a subgroup of $\mathrm{Aut}(C_q^n)$, or equivalently, to a subgroup of $GL_n(q)$. By the proposition above, P is isomorphic to a subgroup of C_p^n. Finally, we have been assuming here that $|Q| = q^b$ and $b \leq 2$, so $n \leq 2$. Thus P is a subgroup of C_p^2, so we are done by Lemma 2.1.

So we have left the cases where $p^3 \leq p^a < q^b$, and $p^a < 51$, and either $p = 3$ and $q = 2$ or $p = 2$ and $q = 3$.

Case 1: Assume $p = 3$ and $q = 2$. Then $3^a = p^a < 51$ and $a \geq 3$ imply $p^a = 3^3$. Then $q^b < 2592/(3^3) = 96$, so $q^b \mid 64 = 2^6$ and $b \leq 6$. But $2^b = q^b > p^a = 27$ by assumption, so $2^b > 27$ and so $b \geq 5$. Thus $(p^a, q^b) \in \{(3^3, 2^6), (3^3, 2^5)\}$.

Case 2: On the other hand, assume $p = 2$ and $q = 3$. Then $p^a = 2^a < 51$ so $p^a \mid 32 = 2^5$. Also, $p^a \geq 2^3$ implies $q^b \leq 2592/2^3 = 324 < 729 = 3^6$, so $q^b \mid 3^5 = 243$.

So in Case 2, $2^3 \mid p^a \mid 2^5$ and $p^a < q^b \mid 3^5$. Our "index case" group has $p^a = 2^5$ and $q^b = 3^4$, so we may eliminate from consideration the cases $(p^a, q^b) = (2^5, 3^4)$, $(p^a, q^b) = (2^5, 3^5)$, and $(p^a, q^b) = (2^4, 3^5)$, as we are looking for groups which are strictly smaller than the one we have already found (clearly $2^4 * 3^5 > 2^5 * 3^4$).

However, if $p^a = 2^5$ then $3^b = q^b > p^a = 2^5 = 32$ implies $b \geq 4$. Therefore we may assume in Case 2 that $a \leq 4$, since we have eliminated the cases where $a = 5$ and $b \geq 4$. Thus the remaining cases are:

(From Case 1) $(p^a, q^b) \in \{(3^3, 2^5), (3^3, 2^6)\}$, and

(From Case 2) $(p^a, q^b) \in \{(2^4, 3^4), (2^4, 3^3), (2^3, 3^5), (2^3, 3^4), (2^3, 3^3)\}$.

Let's start with Case 1, and consider groups of order $3^3 * 2^5 = 864$, this time at $p = 3$.

```
gap> all:=AllSmallGroups(3^3*2^5);;
gap> temp:=[];;
gap> for i in all do
> if IsElementaryAbelian(SylowSubgroup(i,3)) then
> if Size(PCore(i,3)) = 1 then
> Add(temp,i);
> fi;fi;od;
gap> Size(temp);
0
```

So there are no groups of order 864 whose Steinberg complexes exhibit nonprojective homology. We try the other part of Case 1:

```
gap> all:=AllSmallGroups(3^3*2^6);;
gap> temp:=[];;
gap> for i in all do
> if IsElementaryAbelian(SylowSubgroup(i,3)) then
> if Size(PCore(i,3)) = 1 then
> Add(temp,i);
> fi;fi;od;
gap> Size(temp);
1
```

So there is only one candidate group of order $3^3 * 2^6 = 12^3 = 1728$ at $p = 3$. And we know what this group is: the direct product of three copies of the alternating group on 4 letters has an elementary abelian 3-subgroup and has trivial 3-core (and has order 12^3). GAP has told us that it is the only such group.

But $\mathcal{A}_3(A_4 \times A_4 \times A_4)$ is homotopy-equivalent to the join of 3 copies of $\mathcal{A}_3(A_4)$, and $\mathcal{A}_3(A_4)$ is discrete. Thus $\mathcal{A}_3(A_4 \times A_4 \times A_4)$ is a wedge of 2-spheres, so it has reduced homology only in one degree, and so the homology of the Steinberg complex of this group at $p = 3$ is projective. (Since we are taking coefficients in $\mathbb{F}_p$ or some other ring, we are implicitly using the Universal Coefficient Theorem for homology, and the fact that the homology groups of a sphere are torsion-free.)

So we are left to consider Case 2 above. We start with the smallest possibilities: Suppose $|G| = 2^3 * 3^3 = 216$, and let $p = 2$. We can assume as before that the 2-core of G is trivial, and (since $a = 3$) that a Sylow 2-subgroup of G is elementary abelian.

```
gap> all:=AllSmallGroups(216);;
gap> temp:=[];;
gap> for i in all do
> if IsElementaryAbelian(SylowSubgroup(i,2)) then
> if Size(PCore(i,2))=1 then
> Add(temp,i);
> fi;fi;od;
gap> Size(temp);
1
```

Again we have only one candidate, and again we can identify this group: $G = S_3 \times S_3 \times S_3$. Thus $\mathcal{A}_2(G) = \mathcal{A}_2(S_3 \times S_3 \times S_3) = \mathcal{A}_2(S_3) \star \mathcal{A}_2(S_3) \star \mathcal{A}_2(S_3)$, which is a wedge of 2-spheres. Thus it has reduced homology only in dimension 2, so the Steinberg complex has projective homology in every dimension.

So far we have been (apparently) extremely lucky: there have been relatively few group orders which we could not immediately discard, and so far we have had to check at most one group of each order. Probably the best explanation for this is that we have been checking only relatively small groups, and we have been interested mainly in boundary cases (orders which are very close to orders which we could discard). Still, it is remarkable that there has been for each order so far at most one group to consider.

We continue, with groups of order $2^4 * 3^3 = 432$:

```
gap> all:=AllSmallGroups(2^4*3^3);;
gap> temp:=[];;
gap> for i in all do
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;
> od;
gap> Size(temp);
9
gap> temp2:=[];;
gap> for i in temp do
> reps:=ConjugacyClassesSubgroups(SylowSubgroup(i,2));
> reps:=List(reps,Representative);;
> max:=0;
> for j in reps do
> if Size(j) > max then
> if IsElementaryAbelian(j) then
> max:=Size(j);
> fi;fi;od;
> if max > 4 then
> Add(temp2,i);
> fi;
> od;
gap> Size(temp2);
1
gap> temp2;
[ <pc group of size 432 with 7 generators> ]
```

So we have, again, isolated a unique group to study: it happens that there is a unique group of order 432 with trivial 2-core and an elementary abelian 2-subgroup of order 8. Again, we can identify this group: $(S_3 \wr C_2) \times S_3$ has these properties. It is not totally obvious that $O_2((S_3 \wr C_2) \times S_3) = 1$, but it is true: let $G = (S_3 \wr C_2) \times S_3$. Now $O_2(G)$ is the intersection of the Sylow 2-subgroups of G, and a Sylow 2-subgroup of a direct product is a direct product of Sylow 2-subgroups of the factor groups. Of course the Sylow 2-subgroups of S_3 have trivial intersection, so the claim reduces to showing that $O_2(S_3 \wr C_2) = 1$.

If we write $C_2 = \langle \rho \rangle$ and write $S_3 \wr C_2 = (S_3 \times S_3) \rtimes C_2$, then two Sylow 2-subgroups of this group are $\langle ((1,2), e), (e, (1,2)), \rho \rangle$, and $\langle ((1,3), e), (e, (1,3)), \rho \rangle$. Their intersection is $\langle \rho \rangle$, so $O_2(G) \leq \langle \rho \rangle = C_2$, and C_2 is normal in G if and only if it is central in G, and it is certainly not central in G. This proves that $O_2(G) = 1$. (Of course, we could also have shown this in GAP.)

Now, $\mathcal{A}_2(G) \simeq \mathcal{A}_2(S_3 \wr C_2) \star \mathcal{A}_2(S_3) \simeq \mathcal{A}_2(S_3 \wr C_2) \star (\bigvee_{i=1,2} S^0) \simeq \bigvee_{i=1,2} \mathcal{A}_2(S_3 \wr C_2) \star S^0 \simeq \bigvee_{i=1,2} \Sigma \mathcal{A}_2(S_3 \wr C_2)$, and $\tilde{H}_{n+1} \Sigma(\mathcal{A}_2(S_3 \wr C_2)) \cong \tilde{H}_n(\mathcal{A}_2(S_3 \wr C_2)$ for all n. Finally $\tilde{H}_n(\mathcal{A}_2(S_3 \wr C_2)) = 0$ for all $n \neq 1$, because we claim that $\mathcal{A}_2(S_3 \wr C_2)$ is a connected graph: $\mathcal{A}_2(S_3 \wr C_2)$ is clearly a graph since a Sylow 2-subgroup has order 2^3 and is not elementary abelian.

By [33], $\mathcal{A}_2(S_3 \wr C_2)$ is disconnected if and only if $S_3 \wr C_2$ has a strongly-embedded 2-subgroup, which again is a proper subgroup M such that $P \leq M < S_3 \wr C_2$ for some Sylow 2-subgroup P of $S_3 \wr C_2$, and such that $M \cap M^x$ has odd order for all $x \notin M$. It happens that there is no subgroup of order 24 in $S_3 \wr C_2$:

```
gap> w:=WreathProduct(SymmetricGroup(3),Group((1,2)));
Group([ (1,2,3), (1,2), (4,5,6), (4,5), (1,4)(2,5)(3,6) ])
gap> reps:=ConjugacyClassesSubgroups(w);;
gap> reps:=List(reps,Representative);;
gap> Collected(List(reps,Size));
[ [ 1, 1 ], [ 2, 3 ], [ 3, 2 ], [ 4, 3 ], [ 6, 6 ], [ 8, 1 ], [ 9, 1 ],
  [ 12, 2 ], [ 18, 3 ], [ 36, 3 ], [ 72, 1 ] ]
```

Hence the only candidates for such an M would be the Sylow 2-subgroups themselves, and we have seen already an example of an intersection of two of these which has order 2. Thus $\mathcal{A}_2(S_3 \wr C_2)$ is a connected graph. Finally we remark that, as above, the integral homology groups of a graph are torsion-free. Thus the Steinberg complex of $G = (S_3 \wr C_2) \times S_3$ at $p = 2$ has nonzero homology in only one degree, where it is of course projective.

The remaining cases are $(p^a, q^b) \in \{(2^4, 3^4), (2^3, 3^5), (2^3, 3^4)\}$.

```
gap> all:=AllSmallGroups(2^3*3^4);;
gap> temp:=[];;
gap> for i in all do
> if IsElementaryAbelian(SylowSubgroup(i,2)) then
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;fi;od;
gap> Size(temp);
7
gap> all:=AllSmallGroups(2^3*3^5);;
gap> temp:=[];;
gap> for i in all do
> if IsElementaryAbelian(SylowSubgroup(i,2)) then
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;fi;od;
gap> Size(temp);
40
gap> all:=AllSmallGroups(2^4*3^4);;
gap> temp:=[];;
gap> for i in all do
> if Size(PCore(i,2)) = 1 then
> Add(temp,i);
> fi;od;
gap> Size(temp);
68
gap> temp2:=[];;
gap> for i in temp do
> s:=SylowSubgroup(i,2);
> reps:=ConjugacyClassesSubgroups(s);
> reps:=List(reps,Representative);;
> count:=0;
> for j in reps do
> if Size(j) = 8 then
> if IsElementaryAbelian(j) then
> count:=1;
> fi;fi;od;
> if count=1 then
> Add(temp2,i);
> fi;
```

```
> od;
gap> Size(temp2);
14
```

So together these groups make up $7 + 40 + 14 = 61$ isomorphism classes which we must evidently check by hand. (For each of these groups G, $\mathcal{B}_2(G)$ has dimension ≥ 2.)

```
gap> Collected(List(temp2,x->IsElementaryAbelian(SylowSubgroup(x,2))));
[ [ true, 1 ], [ false, 13 ] ]
```

On the plus side, only one of these groups has an elementary abelian subgroup of order 2^4 – all others have rank 3. We can identify this group as $S_3 \times S_3 \times S_3 \times S_3$, whose 2-subgroups complex is a wedge of spheres and has projective $\mathbb{F}_2$-homology. So we have 60 groups left to check, and for each of these it suffices to check the projectivity of $\tilde{H}_2(\mathcal{A}_2(G)) = \mathrm{Ker}(d_2)$ (as we did above for a group of order $2^5 * 3^3 = 864$). In view of the number of groups left to consider, the lack of good qualitative information about these groups, and the extreme lack of variety in the calculations to be performed, it seems to make sense to automate these calculations. The program files used below may be found in the appendices, except for “reps” which may be found at [56]. The file “candidategroups” is simply a list of the 60 groups we had left to check.

```
gap> Read("reps");
gap> Read("psubgroups");
gap> Read("projectivehomology3");
gap> Read("candidategroups");
gap> Size(candidates);
60
gap> PrintTo("projectivehomologyresults","results:=[];");
gap> for ng in [1..60] do
> newresult:=isproj2homology(candidates[ng]);
> AppendTo("projectivehomologyresults","Add(results,");
> AppendTo("projectivehomologyresults",newresult);
> AppendTo("projectivehomologyresults",");");
> od;
gap> Read("projectivehomologyresults");
gap> Collected(results);
[ [ true, 60 ] ]
```

We remark, finally, that for each of these 60 groups we have only checked the projectivity of $\tilde{H}_2(\mathcal{A}_2(G);\mathbb{F}_2)$. However, this is enough, in light of the following two theorems:

Theorem 11.7 *Let A be a ring such that projective modules and injective modules are the same. Let P_* be a complex of A-modules*

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

where P_i is projective for each i. Assume that $Ker(d_2)$ and $H_0(P_*)$ are projective. Then $H_i(P_*)$ is projective for all i.

Proof: Of course $H_i(P_*) = 0$ is projective for $i < 0$ and for $i > 2$, and $H_0(P_*)$ and $H_2(P_*) = \mathrm{Ker}(d_2)$ are projective by assumption. We need to show that $H_1(P_*)$ is projective.

We have a short exact sequence

$$0 \to \mathrm{Im}(d_1) \to P_0 \to P_0/\mathrm{Im}(d_1) \to 0,$$

and by assumption $P_0/\mathrm{Im}(d_1) = H_0(P_*)$ is projective. Therefore the sequence splits and $\mathrm{Im}(d_1)$ is a summand of the projective module P_0; hence $\mathrm{Im}(d_1)$ is projective. This implies that the short exact sequence

$$0 \to \mathrm{Ker}(d_1) \to P_1 \to \mathrm{Im}(d_1) \to 0,$$

splits, and $\mathrm{Ker}(d_1)$ is also projective.(*)

Next, by assumption $\mathrm{Ker}(d_2)$ is projective and therefore injective, so the short exact sequence

$$0 \to \mathrm{Ker}(d_2) \to P_2 \to \mathrm{Im}(d_2) \to 0,$$

splits, and $\mathrm{Im}(d_2)$ is projective and therefore injective. This implies that the short exact sequence

$$0 \to \mathrm{Im}(d_2) \to \mathrm{Ker}(d_1) \to H_1(P_*) \to 0$$

splits, and thus $H_1(P_*)$ is projective by (*). □

Theorem 11.8 *Suppose G has p-rank 3, meaning that a maximal elementary abelian subgroup has size p^3, and $\tilde{H}_2(\mathcal{A}_p(G);\mathbb{F}_p)$ is projective. Then for every field k of characteristic p, $\tilde{H}_n(\mathcal{A}_p(G);k)$ is projective for all n.*

Proof:

By [55] we know that $\tilde{H}_0(\mathcal{A}_p(G);\mathbb{F}_p)$ is projective, so by Theorem 11.7, $\tilde{H}_n(\mathcal{A}_p(G);\mathbb{F}_p)$ is projective for all n.

Let k be a field of characteristic p. By the Universal Coefficients Theorem,

$$\tilde{H}_2(\mathcal{A}_p(G);k) = k \otimes \tilde{H}_2(\mathcal{A}_p(G);\mathbb{Z}),$$

since the complex has dimension 2 and so the degree-2 integral homology is a free abelian group and thus torsion-free. But then

$$\begin{aligned} k \otimes_{\mathbb{Z}} \tilde{H}_2(\mathcal{A}_p(G);\mathbb{Z}) &= (k \otimes_{\mathbb{F}_p} \mathbb{F}_p) \otimes \tilde{H}_2(\mathcal{A}_p(G);\mathbb{Z}) \\ &= k \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes \tilde{H}_2(\mathcal{A}_p(G);\mathbb{Z})) \\ &= k \otimes_{\mathbb{F}_p} \tilde{H}_2(\mathcal{A}_p(G);\mathbb{F}_p). \end{aligned}$$

So let $P = \tilde{H}_2(\mathcal{A}_p(G);\mathbb{F}_p)$, and let Q be such that $P \oplus Q = \mathbb{F}_p G^n$. Then $k \otimes P \oplus k \otimes Q = k \otimes_{\mathbb{F}_p} (P \oplus Q) = k \otimes \mathbb{F}_p G^n = kG^n$. So $k \otimes P$ is projective. $\square$

So in this section we have proven the following:

Theorem 11.9 *Suppose p and q are primes, and suppose J is a group of order $p^a q^b$ for some positive integers a, b. If $p^a q^b < 2^5 * 3^4$, then the Steinberg complex of J, at any prime r, over any r-complete local ring R, will have homology which in every dimension is a projective RJ-module.*

$\square$

Chapter 12

A further look at coefficient systems

We conclude by examining the properties of coefficient systems, and representations of categories more generally. Recall that a representation of a category $\mathcal{C}$ over a ring R is a functor from $\mathcal{C}$ to the category of R-modules. In section 3 we said that a *coefficient system* is a contravariant functor, from the category of G-sets with stabilizers in a certain collection $\mathcal{W}$ of subgroups of G, to the category R-mod of finitely-generated R-modules.

Later on this section we will write $\mathcal{O}$ to mean the full subcategory of the category of G-sets, whose objects are G-sets of the form G/H for some subgroup H. This is the *orbit category* of G, and is equivalent to the category of all transitive G-sets. For now, however, we will work in a more general setting, where we do not use special poset categories $\mathcal{W}$ or $\mathcal{O}$, but any small category $\mathcal{C}$.

Note: in the following discussion, where the word "functor" is used without descriptor, it should be taken to mean "covariant functor." When it comes time to consider the special case of coefficient systems, we will be replacing $\mathcal{C}$ not by $\mathcal{W}$ or $\mathcal{O}$, but by $\mathcal{W}^{op}$ or $\mathcal{O}^{op}$.

Definition 12.1 *Let $\mathcal{C}$ be a small category, and R a commutative ring with identity. If x is an object of $\mathcal{C}$, then we write ${}^xF = RHom_{\mathcal{C}}(x, -)$, where for each object y of $\mathcal{C}$, $RHom_{\mathcal{C}}(x, y)$ is defined to be the free R-module with basis the set of morphisms from x*

to y in $\mathcal{C}$. If $f \in Hom_{\mathcal{C}}(a,b)$, then we define ${}^xF(f) : {}^xF(a) \to xF(b)$, or equivalently

$$ {}^xF(f) : RHom_{\mathcal{C}}(x,a) \to RHom_{\mathcal{C}}(x,b), $$

by giving its values on generators, according to the rule

$$ (\alpha : x \to a) \mapsto (f \circ \alpha : x \to b). $$

Thus for each object x of $\mathcal{C}$, we get a covariant functor ${}^xF : \mathcal{C} \to R$-mod. However, a morphism $h \in \mathrm{Hom}_{\mathcal{C}}(y,z)$ induces a natural transformation $\eta_h : {}^zF \to {}^yF$, or equivalently

$$ \eta_h : R\mathrm{Hom}_{\mathcal{C}}(z,-) \to R\mathrm{Hom}_{\mathcal{C}}(y,-), $$

in the following way: at each object a of $\mathcal{C}$, we have that η_h^a is given on generators by

$$ (\alpha : z \to a) \mapsto (\alpha \circ h : y \to a). $$

So ${}^-F$ is a contravariant functor from $\mathcal{C}$ to the category (R-mod$^{\mathcal{C}}$), which has for its objects the functors $\mathcal{C} \to R$-mod and for its morphisms the natural transformations between these functors.

Proposition 12.2 *(Webb, Proposition 4.4 in [54]) Let $\mathcal{C}$ be a small category with finitely many objects. Then:*

(1) If $M : \mathcal{C} \to R$-mod is a functor, then $Hom_{R\mathcal{C}}({}^xF, M) \cong M(x)$ as R-modules. (We may think of representations of $\mathcal{C}$ over R as $R\mathcal{C}$-modules; hence the notation $Hom_{R\mathcal{C}}$.)

(2) ${}^xF$ is a projective $R\mathcal{C}$-module.

□

Remark: The proofs go exactly as they did in section 3 for the special case of coefficient systems. We rely on the assumption that $\mathcal{C}$ be a small category with finitely many objects in order to identify $(R\text{-mod})^{\mathcal{C}}$ with $R\mathcal{C}$-mod, where $R\mathcal{C}$ is the category algebra: namely, the free R-module generated by the set of morphisms of $\mathcal{C}$, with multiplication defined on generators f and g by:

$$ f * g = \begin{cases} f \circ g & \text{if } f \circ g \text{ is defined} \\ 0 & \text{else.} \end{cases} $$

If it happens to be the case that $\mathcal{C}$ has one object and its morphisms form a group G under composition, then $R\mathcal{C}$ is the group algebra RG, a functor $\mathcal{C} \to R$-mod is exactly a representation of G over R, and the the equivalence is well-known. The equivalence in the general case, along with many many other theorems of representation theory of categories, is due to Webb and may be found in [54].

We note by way of example that the category algebra $R\mathcal{C}$ has identity element $\sum_{X \in Ob(\mathcal{C})} \mathrm{Id}_X$. Requiring $\mathcal{C}$ to have only finitely many objects ensures that this sum will be finite.

Definition 12.3 *For any functors $M, N : \mathcal{C} \to R$-mod, we define a functor $(M \otimes N) : \mathcal{C} \to R$-mod by $(M \otimes N)(X) = M(x) \otimes_R N(X)$ for any object X of $\mathcal{C}$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, then $(M \otimes N)(f) : (M \otimes N)(X) \to (M \otimes N)(Y)$ is defined as $M(f) \otimes N(f) : M(X) \otimes N(X) \to M(Y) \otimes N(Y)$.*

We also define $\mathcal{H}om(M, N) : \mathcal{C} \to R$-mod, by $\mathcal{H}om(M, N)(y) := Hom_{R\mathcal{C}}({}^yF \otimes M, N)$, for all objects y of $\mathcal{C}$. Thus if $y \in Ob(\mathcal{C})$ and F, M, N are functors $\mathcal{C} \to R$-mod, then $\mathcal{H}om(M, N)(y)$ is the set of natural transformations from ${}^yF \otimes M$ to N. This set of natural transformations is itself an R-module. Now if $f : x \to y$ is a morphism in $\mathcal{C}$, then f induces maps ${}^yF \to {}^xF$, hence ${}^yF \otimes M \to {}^xF \otimes M$, and finally $\mathcal{H}om(M, N)(x) \to \mathcal{H}om(M, N)(y)$.

We have defined $\mathcal{H}om$ so that we have the following adjoint relation:

Proposition 12.4 *$Hom(L, \mathcal{H}om(M, N)) \cong Hom(L \otimes M, N)$ as R-modules.*

Proof: $\mathrm{Hom}({}^zF \otimes M, N) = \mathcal{H}om(M, N)(z) = \mathrm{Hom}({}^zF, \mathcal{H}om(M, N))$, so the equation holds when $L = {}^zF$, and more generally when $L = \bigoplus {}^{z_i}F$. For a general L we take a projective resolution of L, or indeed a free resolution. We note that $R\mathcal{C} = \bigoplus_{x \in Ob(\mathcal{C})} R\mathcal{C} * \mathrm{Id}_x \cong \bigoplus_{x \in Ob(\mathcal{C})} {}^xF$, so that a free resolution will be made of sums of various ${}^{x_i}F$. Thus:

$$\ldots \to \bigoplus {}^{y_j}F \to \bigoplus {}^{x_i}F \to L \to 0$$

is exact, so

$$0 \to \mathrm{Hom}(L \otimes M, N) \to \mathrm{Hom}((\oplus^{x_i}F) \otimes M, N) \to \mathrm{Hom}((\oplus^{y_j}F) \otimes M, N)$$

is exact. Similarly,

$$0 \to \mathrm{Hom}(L, \mathcal{H}om(M,N)) \to \mathrm{Hom}((\oplus^{x_i} F), \mathcal{H}om(M,N)) \to \mathrm{Hom}((\oplus^{y_j} F), \mathcal{H}om(M,N))$$

is exact. Now, adding in the isomorphisms we have already established, we get the commutative diagram shown in Figure 12, where the vertical maps are both fixed

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(L, \mathcal{H}om(M,N) \arrow[r] & \mathrm{Hom}((\oplus^{x_i} F), \mathcal{H}om(M,N)) \arrow[r] \arrow[d] & \mathrm{Hom}((\oplus^{y_j} F), \mathcal{H}om(M,N)) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(L \otimes M, N) \arrow[r] & \mathrm{Hom}((\oplus^{x_i} F) \otimes M, N) \arrow[r] & \mathrm{Hom}((\oplus^{y_j} F) \otimes M, N),
\end{tikzcd}$$

Figure 12.1: A commutative diagram involving exact sequences

isomorphisms. Then there is a third vertical isomorphism $\mathrm{Hom}(L, \mathcal{H}om(M,N)) \to \mathrm{Hom}(L \otimes M, N)$ which makes the diagram commute. □

Peter Symonds has, for the case of the orbit complex only, a different definition of the $\mathcal{H}om(M,N)$ functor, which also satisfies the same adjointness relation [47]. In that special case, the two definitions are equivalent.

We are often interested in the case where $\mathcal{C} = \mathcal{O}^{op}$, since this is the case of a coefficient system over the collection $\mathcal{O}$ of all subgroups of G.

Proposition 12.5 *When $\mathcal{C} = \mathcal{O}^{op}$, we have ${}^xF \otimes {}^yF = {}^{x \times y}F$.*

Proof: When $\mathcal{C} = \mathcal{O}^{op}$, we have ${}^xF = R[x^?]$. Then

$${}^xF \otimes {}^yF(G/H) = R[x^H] \otimes R[y^H] = R[x^H \times y^H] = R[(x \times y)^H] = {}^{x \times y}F(G/H).\square$$

Definition 12.6 *Let $\mathcal{C} = \mathcal{O}^{op}$. If $M : \mathcal{C} \to R$-mod is a functor (*i.e.*, M is a representation of $\mathcal{C}$ over R), and x is an object of $\mathcal{C}$, then we define*

(i) $M_x(y) = M(y \times x)$, and

(ii) ${}_xM = M \otimes {}^xF$.

(iii) $\underline{R} : \mathcal{C} \to R$-mod is the constant functor*, that is, the functor which sends every object to the free module R and every morphism to the identity.*

Thus $M_x, {}_xM$, and $\underline{R}$ are representations of $\mathcal{C}$.

In particular, ${}_x\underline{R} = {}^xF$, and $M_{G/G} = M = {}_{G/G}M$.

Now, given a map of G-sets $x \to z$ in $\mathcal{O}$, hence $z \to x$ in $\mathcal{C} = \mathcal{O}^{op}$, we get natural transformations $M_z \to M_x$ and ${}_xM \to {}_zM$. Thus if Δ is a G-simplicial chain complex, we get complexes

$$M_\Delta = \ldots \to 0 \to M_{\Delta_{-1}} \to M_{\Delta_0} \to M_{\Delta_1} \to \ldots$$

and

$${}_\Delta M = \ldots \to {}_{\Delta_1}M \to {}_{\Delta_0}M \to {}_{\Delta_{-1}}M \to 0 \to \ldots.$$

Proposition 12.7 *Let $\mathcal{C} = \mathcal{O}^{op}$, and let M and N be representations of $\mathcal{C}$ over R. Then:*

1) $(M_x)_y = M_{x\times y}$.

2) ${}_y({}_xM) = {}_{y\times x}M$

3) ${}_xM \otimes {}_yN = {}_{x\times y}(M \otimes N)$.

4) $\underline{R}_\Delta = \mathcal{H}om({}_\Delta\underline{R}, \underline{R}) = \mathcal{H}om(R[\Delta^?], \underline{R})$.

5) $\mathcal{H}om({}_y\underline{R}, N) = N_y$.

6) $Hom({}_xM, N) = Hom(M, N_x)$.

7) $\mathcal{H}om(M, N_x) = (\mathcal{H}om(M, N))_x$.

Proof:

1) $(M_x)_y(z) = M(z \times x \times y) = M_{x\times y}(z)$. All the equalities are natural isomorphisms.

2) ${}_y({}_xM) = {}^yF \otimes ({}^xF \otimes M) = ({}^yF \otimes {}^xF) \otimes M = {}^{y\times x}F \otimes M = {}_{y\times x}M$.

3) ${}_xM \otimes {}_yN = {}^xF \otimes M \otimes {}^yF \otimes N = {}^xF \otimes {}^xF \otimes M \otimes N = {}^{x\times y}(M \otimes N) = {}_{x\times y}(M \otimes N)$.

4) By (3), taking $N = \underline{R}$, we get $\underline{R}_y = \mathcal{H}om({}_y\underline{R}, \underline{R})$. Thus

$$\underline{R}_\Delta = \mathcal{H}om({}_\Delta\underline{R}, \underline{R}) = \mathcal{H}om(R[\Delta^?], \underline{R}).$$

5) $\mathcal{H}om(M, N)(x) = \mathrm{Hom}({}^xF \otimes M, N)$ for all M, N. Now taking $M = {}^yF = {}_y\underline{R}$, we get $\mathcal{H}om({}^yF, N)(x) = \mathrm{Hom}({}^{x\times y}F, N)$. By the Yoneda lemma, this is $N_y(x)$.

6) $\mathrm{Hom}(M, N_x) = \mathrm{Hom}(M, \mathcal{H}om({}^xF, N)) \cong \mathrm{Hom}(M \otimes {}^xF, N) = \mathrm{Hom}({}_xM, N)$.

7) $\mathcal{H}om(M, N)_x(y) = \mathcal{H}om(M, N)(x \times y) = \mathrm{Hom}({}^{x\times y}F \otimes M, N) = \mathrm{Hom}({}^yF \otimes {}^xF \otimes M, N) = \mathcal{H}om({}^xF \otimes M, N)(y) = \mathcal{H}om({}_xM, N)(y) = \mathcal{H}om(M, N_x)(y)$. $\square$

Theorem 12.8 *(Eilenberg-Zilber, Theorem 3.1 in [28].) Let X and Y be topological spaces. Then there exists a chain homotopy equivalence*

$$\zeta : C(X) \otimes C(Y) \to C(X \times Y)$$

which is natural in X and Y – that is, given continuous maps $f : X \to X'$ and $g : Y \to Y'$, the diagram in Figure 12.8 commutes.

$$\begin{tikzcd}
C(X) \otimes C(Y) \arrow[r, "\zeta"] \arrow[d, "f_\sharp \otimes g_\sharp"] & C(X \times Y) \arrow[d, "(f \times g)_\sharp"] \\
C(X') \otimes C(Y') \arrow[r, "\zeta"] & C(X' \times Y')
\end{tikzcd}$$

Figure 12.2: The Eilenberg-Zilber theorem for Coefficient Systems

We remark that the proof of this theorem does not require taking coefficients in the ring $\mathbb{Z}$; we can take $C(-)$ to be $C_*(-; R)$ for any commutative ring R, by applying the functor $- \otimes R$ everywhere.

Theorem 12.9 *Let G be a finite group, let R be a commutative ring with identity, and let Δ, Θ be G-simplicial complexes. Then there exists a chain homotopy equivalence*

$$R[(\Delta \times \Theta)^?] \simeq R[\Delta^?] \otimes R[\Theta^?].$$

Thus these complexes are isomorphic in the homotopy category of coefficient systems.

Proof: For each $H \leq G$, we have $(\Delta \times \Theta)^H = \Delta^H \times \Theta^H$ as simplicial complexes. Thus $R[(\Delta \times \Theta)^H] = R[\Delta^H \times \Theta^H]$. Therefore by the Eilenberg-Zilber Theorem, $C_*(\Delta \times \Theta)^H; R) \simeq C_*(\Delta^H; R) \otimes C_*(\Theta^H; R)$. Rewriting to change notation, we get a chain homotopy equivalence $\zeta^H : R[(\Delta \times \Theta)^H] \simeq R[\Delta^H] \otimes R[\Theta^H]$.

Finally, we need these equivalences to be natural in H, so that if $H \leq K^g$, then the following diagram commutes:

$$\begin{tikzcd}
R[(\Delta \times \Theta)^H] \arrow[r, "\zeta^H"] & R[\Delta^H] \otimes R[\Theta^H] \\
R[(\Delta \times \Theta)^{K^g}] \arrow[r, "\zeta^{K^g}"] \arrow[u] & R[\Delta^{K^g}] \otimes R[\Theta^{K^g}] \arrow[u]
\end{tikzcd}$$

Fortunately, the inclusion and conjugation maps are continuous, so the naturality condition is guaranteed by the Eilenberg-Zilber Theorem. Finally, we also require that all the chain homotopies involved are natural in X and Y; this is the content of Lemmas 5.7 and 5.8 in [28]. □

Remark: a chain map of complexes $f : F_* \simeq G_*$ of coefficient systems is a doubly-indexed family of R-module maps, $f_n(H) : F_n(H) \to G_n(H)$. Fixing n and letting H vary gives a morphism of coefficient systems $F_n \to G_n$. Fixing H and letting n vary gives a chain map of complexes of R-modules, $F_*(H) \to G_*(H)$. We have chosen to define a chain map of complexes of coefficient systems using the second approach: for each H the Eilenberg-Zilber Theorem guarantees the existence of a certain chain map of complexes of R-modules, and also that the collection of all these maps is natural in H.

Theorem 12.10 *Let G be a finite group, let $\mathcal{O}$ be the set of all subgroups of G, and let $\Delta = \tilde{C}_*(\mathcal{S}_p(G); R)$. If $O_p(G) = 1$ then $End_{R\mathcal{O}^{op}}({}_\Delta R)$ is chain homotopy-equivalent to the complex which is R concentrated in degree 0. If $O_p(G) \neq 1$ then $End_{R\mathcal{O}^{op}}({}_\Delta R)$ is chain homotopy-equivalent to the zero complex.*

Proof: We form the double complex with terms

$$\mathrm{Hom}(\bigoplus_{\sigma \in G\backslash\Delta_i} R[(G/G_\sigma)^?], \bigoplus_{\tau \in G\backslash\Delta_j} R[(G/G_\tau)^?]) = \bigoplus_{\sigma,\tau} R[(G/G_\tau^{G_\sigma})],$$

whose homology is homotopy classes of maps. We filter the double complex so that we get a page of the spectral sequence made up of complexes $R[\Delta^{G_\sigma}]$ for each σ. When $\dim \sigma \geq 0$ this is contractible and gives 0 in homology. When $\dim \sigma = -1$, we have $G_\sigma = G$ (every group element fixes the empty simplex). If another simplex is fixed then it means that some nonidentity p-subgroup of G is normal in G, or in other words

$O_p(G) \neq 1$. In that case $\tilde{\Delta} \simeq_G 0$ and ${}_{\Delta}\underline{R} = 0$. Hence at the next page, when $O_p(G) = 1$, there is only one non-zero term, in position $(-1, -1)$, giving R. $\square$

It would be nice if in fact $\mathcal{H}om({}_{\Delta}\underline{R}, {}_{\Delta}\underline{R})$ were $\underline{R}$. If so, that would suggest the possibility of a duality operator $\mathcal{H}om(-, {}_{\Delta}\underline{R})$ on complexes of coefficient systems, which in particular interchanged ${}_{\Delta}\underline{R}$ and $\underline{R}$.

Or it might signify a "tilting" property that the Steinberg complex in general did not have. Recall that the Steinberg complex was found essentially by taking the complex $R[\Delta^?]$ of coefficient systems, and evaluating in each dimension at the identity subgroup. The resulting complex of RG-modules did not have a desirable tilting property in general: the complex admitted nonzero self-chain maps in positive or negative degree. It may be that it is more desirable not to evaluate at the identity subgroup, since we would in essence then retain more conditions that a chain map has to satisfy, and thus allow fewer chain maps to interfere with the tilting condition.

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Appendix A

GAP routines used in calculations

It has been necessary to use computers to assist in some of the calculations performed in this work. This has been done in GAP. The new software written for the purpose of this work, specifically section 11, is presented here.

```
psubs:=function(g,p)

## This program calculates the different subgroup complexes for a group
## g at a prime p.

local reps,sp,ap,bp,zp,i,elts,x;
reps:=ConjugacyClassesSubgroups(g);
reps:=List(reps,Representative);

sp:=[];
for i in reps do
if Size(i) in DivisorsInt(Size(SylowSubgroup(g,p))) then
if Size(i) > 1 then
Add(sp,i);
fi;fi;od;

ap:=[];
for i in sp do
if IsElementaryAbelian(i) then
Add(ap,i);
fi;od;

zp:=[];
```

```
for i in ap do
elts:=[];
for x in Center(Centralizer(g,i)) do
if Order(x) in [1,p] then
Add(elts,x);
fi;od;
if Group(elts) = i then
Add(zp,i);
fi;od;

bp:=[];
for i in sp do
if PCore(Normalizer(g,i),p) = i then
Add(bp,i);
fi;od;

return([sp,ap,zp,bp]);
end;

##
## Must read "reps" and "psubgroups" first.
##
## More importantly, this function as written only works for
## p=2, and only for groups of 2-rank 3 (a maximal elementary abelian
## 2-subgroup has order 2^3).
##
## This program checks whether or not the homology in dimension 2
## of the Steinberg complex of a group g at the prime 2 over the
## field of 2 elements is projective
##


isproj2homology:=function(g)
 local z2,triangles,i,j,ori,orj,l,m,n,k,ork,lines,delta,null,
   mats,mat,x,rep,count;
 if not 2 in DivisorsInt(Size(g)) then
   return true;
 fi;
 z2:=psubs(g,2)[3];
 triangles:=[];;
 lines:=[];;
 for i in z2 do
```

```
  for j in z2 do
   if Size(j) > Size(i) then
     ori:=ConjugateSubgroups(g,i);
     orj:=ConjugateSubgroups(g,j);
     for l in ori do
       for m in orj do
         if IsSubgroup(m,l) then
           Add(lines,[l,m]);
         fi;
       od;
     od;
     for k in z2 do
       if Size(k) > Size(j) then
 ork:=ConjugateSubgroups(g,k);
 for l in ori do
   for m in orj do
     for n in ork do
if IsSubgroup(m,l) and IsSubgroup(n,m) then
  Add(triangles,[l,m,n]);
fi;
     od;
   od;
 od;
       fi;
     od;
   fi;
  od;
 od;
 delta:=NullMat(Size(triangles),Size(lines),GF(2));
 for i in [1..Size(triangles)] do
   j:=1;
   count:=0;
   while count < 3 do
     if IsSubset(triangles[i],lines[j]) then
       delta[i][j]:=Z(2)^0;
       count:=count+1;
     fi;
     j:=j+1;
   od;
 od;
 null:=NullspaceMat(delta);;
```

```
 if not Size(SylowSubgroup(g,2)) in DivisorsInt(Length(null)) then
   return false;
 fi;
 mats:=[];
 for x in GeneratorsOfGroup(g) do
   mat:=NullMat(Size(triangles),Size(triangles),GF(2));
   for i in [1..Size(triangles)] do
     j:=1;
     while not OnTuples(triangles[i],x) = triangles[j] do
       j:=j+1;
     od;
     mat[i][j]:=Z(2)^0;
   od;
   Add(mats, mat);
 od;
 rep:=Rep(g,mats);
 if not Length(null)=Length(Spin(rep,null)) then
   return fail;
 fi;
 null:=SubmoduleRep(rep,null);
 return IsProjectiveRep(null);
end;
```