EXTENDING THE COINVARIANT THEOREMS OF CHEVALLEY, SHEPHARD–TODD, MITCHELL, AND SPRINGER

ABRAHAM BROER, VICTOR REINER, LARRY SMITH, AND PETER WEBB

Abstract. We extend in several directions invariant theory results of Chevalley, Shephard and Todd, Mitchell and Springer. Their results compare the group algebra for a finite reflection group with its coinvariant algebra, and compare a group representation with its module of relative coinvariants. Our extensions apply to arbitrary finite groups in any characteristic.

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Let k be an arbitrary field, V an n-dimensional k-vector space, and G a finite subgroup of $GL(V)$. Then G acts by linear change of coordinates on the algebra $k[V]$ of polynomial functions on V. If $x_1, \ldots, x_n$ is a basis for the dual vector space V^* we may identify $k[V]$ with the polynomial algebra $k[x_1, \ldots, x_n]$ in $x_1, \ldots, x_n$ regarded as formal variables. The **coinvariant algebra** is the quotient algebra

$$k[V]/(k[V]^G_+) \cong k[V] \otimes_{k[V]^G} k$$

where $(k[V]^G_+)$ denotes the ideal of $k[V]$ generated by the elements of the invariant subalgebra $k[V]^G$ of strictly positive degree, and $k = k[V]^G/k[V]^G_+$ is regarded as a trivial $k[V]^G$-module. The coinvariant algebra is a finite dimensional G-representation. Much of its significance (see, e.g., [4], [13], [25]) derives from the fact that in favorable cases it provides a *graded* version of the regular representation $k(G)$. This is made precise in the following result, due to Chevalley (and also Shephard and Todd) in characteristic zero, and to Mitchell in positive characteristic.

Theorem(Chevalley [7]**, Shephard and Todd** [24]**, Mitchell** [17]**).** *Let k be an arbitrary field, V an n-dimensional k-vector space, and G a finite subgroup of $GL(V)$. Suppose that $k[V]^G$ is a polynomial algebra. Then as $k(G)$-modules, the coinvariant algebra $k[V] \otimes_{k[V]^G} k$ and the regular representation $k(G)$ have the same composition factors counting multiplicities.*

Something similar for **relative invariants** may be deduced under suitable hypotheses as we explain next. Suppose we have a second group Γ and a $(k(\Gamma), k(G))$-bimodule U, i.e., a right $k(G)$-module U which has a commuting left action of $k(\Gamma)$. We chose to use the terminology of bimodules here as an aid in keeping the different actions distinct. If however no confusion can arise we frequently identify $(k(\Gamma), k(G))$-bimodules with left $k(\Gamma \times G)$-modules; the left action of $(\gamma, g) \in \Gamma \times G$ on an element $u \in U$ being given by $\gamma u g^{-1}$. The module M of U-**relative invariants** is defined by $M := (U \otimes_k k[V])^G$ where G acts on the tensor product diagonally, viz., $g(u \otimes x) = ug^{-1} \otimes gx$. Note that M has the structure of a graded $(k(\Gamma), k[V]^G)$-bimodule.

Corollary. *Let k be a field, $V = k^n$ an n-dimensional vector space over k, G a finite subgroup of $GL(V)$, Γ a second finite group, and U a finite-dimensional $(k(\Gamma), k(G))$-bimodule. Regard the relative invariants*

$$M = (U \otimes_k k[V])^G$$

as a $(k(\Gamma), k[V]^G)$-bimodule. If $|G|$ is invertible in k and $k[V]^G$ is polynomial algebra, then one has a $k(\Gamma)$-module isomorphism

$$M \otimes_{k[V]^G} k \cong U.$$

Note that this corollary includes the nonmodular version of the Chevalley, Shephard-Todd, Mitchell Theorem as the special case where $\Gamma = G$ and $U = k(G)$ as a $(k(G), k(G))$-bimodule. More generally, if we let H be any subgroup of G and $\Gamma = N_G(H)$, the normalizer of H in G, we obtain a k-linear representation $U = k(H\backslash G)$ from the permutation representation of G on the set $H\backslash G$ of right cosets of H in G. Regarded as a $(k(\Gamma), k(G))$-bimodule, the relative invariants $M = (U \otimes_k k[V])^G$ become the subalgebra of H-invariant polynomials $k[V]^H$ regarded as a $k(N_G(H))$-module (see §3.6 below). Certain other cases of relative invariant modules M appear frequently in the literature, such as the i^{th}-exterior power $U = \wedge^i(V^*)$ of V^* (resp. $U = V$), and M is the module of *G-invariant differential i-forms* (resp. *G-invariant vector fields*) on V, (see e.g., [28], [19, §6.1]), or, for a simple $k(G)$-module U where the Hilbert series of $M \otimes_{k[V]^G} k$ defines the *fake degrees* for U (see e.g., [11, §1.6]).

Our first main result, Theorem 1.1.1 below, extends the Chevalley, Shephard–Todd, Mitchell result and its corollary by removing the hypothesis that $k[V]^G$ be polynomial and $|G|$ lie in $k^\times$. Our other main results are inspired by a generalization of the Chevalley-Shephard-Todd Theorem due to Springer [29], which incorporates the action of an extra cyclic group. We recall this next.

Given a finite subgroup $G \subseteq GL(V)$, say that $v \in V$ is a **regular vector** if the orbit Gv is a **regular orbit**, meaning that the stabilizer in G of v is 1, or equivalently that the orbit achieves the maximum cardinality $|Gv| = |G|$. An element $c \in G$ is a **regular element** if it has a regular eigenvector $v \in V$, after possibly extending the field k to include the corresponding eigenvalue $\omega \in k^\times$. Letting $C = \langle c \rangle$ denote the cyclic subgroup generated by c, the group algebra $k(G)$ becomes a $(k(G), k(C))$-bimodule in which (g, c) acts on the basis element t_h of $k(G)$ corresponding to h in G via $(g, c) \cdot t_h := t_{ghc}$. As before we may identify $(k(G), k(C))$-bimodules with $k(G \times C)$-modules by letting a left action of $c \in C$ correspond to a right action of c^{-1}. We also let C act on $k[V]$ by the algebra automorphisms which are a scalar multiplication in each degree and determined by requiring $c^j(x_i) = \omega^j x_i$ for $i = 1, 2, \ldots, n$. In this way we obtain the structure of a $(k(G), k(C))$-bimodule on $k[V]$ as well as $k[V] \otimes_{k[V]^G} k$. The following theorem was proven by Springer in characteristic zero, and extended to arbitrary fields in [20].

Theorem(Springer [29]**, Reiner-Stanton-Webb** [20]**).** *Let k be a field, V a finite-dimensional k-vector space, G a finite subgroup $G \subset GL(V)$. Suppose that $k[V]^G$ is a polynomial algebra, and c is a regular element of G.*

Let $C = \langle c \rangle$ as above. Then one has the equality

$$\left[k[V] \otimes_{k[V]^G} k\right] = [k(G)].$$

in $\mathbf{R}(k(G \times C))$, where $\mathbf{R}(k(G \times C))$ denotes the Grothendieck ring of finite dimensional $(k(G), k(C))$-bimodules.

We are concerned with extensions of these results to arbitrary groups in any characteristic. This will require significant reformulation since the naive versions of these results would not be correct. For example, a simple consequence of the Chevalley, Shephard–Todd, Mitchell result is that the coinvariant algebra for G has dimension $|G|$ whenever $k[V]^G$ is a polynomial algebra, but this fails when $k[V]^G$ is not polynomial (see e.g., [26]).

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1. Statement of Results

In this section we state our main results and illustrate them with a simple example. For background material on invariant theory see [4, 8, 25], on representation theory see [23], particularly Part III, and on reflection groups see [11, 13].

1.1. **Chevalley, Shephard–Todd, Mitchell Type Results.** We first indicate how to remove the hypothesis that $k[V]^G$ be polynomial, and $|G|$ lie in $k^\times$, from the Chevalley, Shephard–Todd, Mitchell Theorem and its corollary. Our result, Theorem 1.1.1, compares the ungraded $k(\Gamma)$-module U with various graded $k(\Gamma)$-modules, showing how, in a sense to be made precise below,

(i) $M \otimes_{k[V]^G} k$ is an overestimate for U,
(ii) it is the first in a sequence of alternating overestimates and underestimates, and
(iii) these estimates converge to a suitably defined limit.

To explain what this means, recall that $k(\Gamma)$ is not in general semisimple, so one compensates for this by working with composition factors. A convenient way to do this is to introduce the **Grothendieck ring** $\mathbf{R}(k(\Gamma))$ of finite-dimensional $k(\Gamma)$-modules. This is defined to be the ring with one generator $[M]$ for each isomorphism class $\{M\}$ of finite dimensional $k(\Gamma)$-modules, and one relation $[M'] - [M] + [M''] = 0$ for each short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Addition in $\mathbf{R}(k(\Gamma))$ is induced by direct sum and product by tensor product. There is also a partial ordering defined by requiring for two elements $x, y \in \mathbf{R}(k(\Gamma))$ that $x \geq y$ if $x - y = [M]$ for some (genuine) $k(\Gamma)$-module M. For example, given two genuine modules M_1, M_2, the inequality $[M_1] \geq [M_2]$ means that for every simple $k(\Gamma)$-module S one has the inequality of composition multiplicities $[M_1 : S] \geq [M_2 : S]$, where $[M : S]$ denotes the number of occurrences of the simple module S as a composition factor in M.

Given a finite group Γ, a (non-negatively) **graded** $k(\Gamma)$-module is one with a direct sum decomposition $M = \oplus_{d \geq 0} M_d$ in which each M_d is a finite-dimensional $k(\Gamma)$-module. Such an M gives rise to an element $[M](t) := \sum_{d \geq 0} [M_d] t^d$ in the formal power series ring over the Grothendieck ring $\mathbf{R}(k(\Gamma))[[t]] := \mathbb{Z}[[t]] \otimes_{\mathbb{Z}} \mathbf{R}(k(\Gamma))$. If one forgets the group action we obtain the formal power series $\sum_{d \geq 0} \dim_k(M_d) t^d \in \mathbb{Z}[[t]]$ often called the **Hilbert series** of M.

The motivating example for us of such a graded $k(\Gamma)$-module will be

$$\oplus_{i \geq 0} \mathrm{Tor}_i^R(M, k)$$

in the situation where R is a finitely generated graded, connected, commutative k-algebra with a grade-preserving action of Γ, and M is a finitely generated graded R-module with a compatible $k(\Gamma)$-module structure (see Section 2). These hypotheses imply that $\mathrm{Tor}_i^R(M, k)$ acquires a grading from R and M, and that each graded component $\mathrm{Tor}_i^R(M, k)_j$ is finite-dimensional over k. Furthermore, for each fixed i this component is non-zero for only finitely many j and for each fixed j the component is non-zero for only finitely many i. This means that $\sum_j (-1)^i [\mathrm{Tor}_i^R(M, k)_j] t^j$ is in fact a polynomial in t for each fixed i. We will also consider the infinite alternating sum $\sum_{i \geq 0} (-1)^i \; [\mathrm{Tor}_i^R(M, k)](t)$, which as a consequence of these finiteness properties is a well-defined element in $\mathbf{R}(k(\Gamma))[[t]]$. This sum generalizes the multiplicity symbol of Serre from the case with no group action (see e.g., [22] and [27, §3]).

Theorem 1.1.1. *Let k be a field, V a finite-dimensional k-vector space, G a finite subgroup of $GL(V)$, and Γ a finite group. Let U be a finite-dimensional $(k(\Gamma), k(G))$-bimodule, and let $M := (U \otimes_k k[V])^G$, regarded as a $(k(\Gamma), k[V]^G)$-bimodule. Set $K := k(V)^G$, the field of G-invariant rational functions on V (the ungraded field of fractions of $k[V]^G$).*

(i) In $\mathbf{R}(k(\Gamma))$, one has the inequality

$$[M \otimes_{k[V]^G} k] \geq [U].$$

Furthermore, one has equality if and only if M is $k[V]^G$-free, in which case $K \otimes_k U$ has a $K(\Gamma)$-module filtration $\{\mathcal{F}_j\}$ for which the factor $\mathcal{F}_j / \mathcal{F}_{j-1}$ is isomorphic to the j^{th} homogeneous component $(M \otimes_{k[V]^G} k)_j \otimes_k K$.

(ii) More generally, for any $m \geq 0$, in $\mathbf{R}(k(\Gamma))$, one has the inequality

$$\sum_{i=0}^{m} (-1)^i \; \left[\mathrm{Tor}_i^{k[V]^G}(M, k) \right] \begin{cases} \geq [U] & \text{if } m \text{ is even,} \\ \leq [U] & \text{if } m \text{ is odd,} \end{cases}$$

with equality if and only if $\mathrm{Tor}_i^{k[V]^G}(M, k)$ vanishes for $i > m$.

(iii) The element $\sum_{i=0}^{\infty}(-1)^i \left[\mathrm{Tor}_i^{k[V]^G}(M,k)\right](t)$ *of* $\mathbf{R}(k(\Gamma))[[t]]$ *lies in the subring* $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \mathbf{R}(k(\Gamma))$ *where each simple module* S *has composition multiplicity series lying in* $\mathbb{Q}(t)$*. Furthermore,* $t = 1$ *is a regular value of this rational function and*

$$\sum_{i \geq 0}(-1)^i \left[\mathrm{Tor}_i^{k[V]^G}(M,k)\right](t) \Bigg|_{t=1} = [U] \quad \text{in } \mathbf{R}(k(\Gamma)).$$

Theorem 1.1.1 is proven in Section 2.5, using a homological strengthening of Chevalley's method from [7], relying ultimately on the Normal Basis Theorem from Galois theory. We illustrate it here with a simple example.

Example 1. Let G be the cyclic group $\mathbb{Z}/2$ of order 2 regarded as the subgroup of $GL(2,\mathbb{C})$ generated by the scalar matrix $g = -I_{2\times 2}$ which is the negative of the identity. Note that the ring of invariants is not a polynomial algebra, rather

$$R := \mathbb{C}[x,y]^{\mathbb{Z}/2} = \mathbb{C}[x^2, xy, y^2].$$

There are two simple $\mathbb{C}(\mathbb{Z}/2)$-modules, U_+ and U_-, both 1-dimensional, with g acting by the scalar $+1, -1$ on U_+, U_-, respectively.

Choose Γ to be the trivial subgroup $\{1\}$ of $\mathrm{Aut}_{\mathbb{C}(\mathbb{Z}/2)}(U_\pm)$. In this case, the Grothendieck ring $\mathbf{R}(\mathbb{C}(\Gamma))$ is isomorphic to $\mathbb{Z}$, with the isomorphism sending the class $[\mathbf{1}]$ of the trivial 1-dimensional $\mathbb{C}(\Gamma)$-module to the integer 1. Any $\mathbb{C}$-vector space of dimension d then represents the element $[\mathbb{C}^d] = d[\mathbf{1}]$ in $\mathbf{R}(\mathbb{C}(\Gamma))$. In particular, $[U_+] = [U_-] = [\mathbf{1}]$ in $\mathbf{R}(\mathbb{C}(\Gamma))$.

One can easily check that in this case

$$M_+ := (U_+ \otimes_{\mathbb{C}} \mathbb{C}[x,y])^{\mathbb{Z}/2} = \mathbb{C}[x,y]^{\mathbb{Z}/2} = R$$
$$M_- := (U_- \otimes_{\mathbb{C}} \mathbb{C}[x,y])^{G} = Rx + Ry.$$

Here M_+ is a free R-module of rank 1, and all inequalities asserted in Theorem 1.1.1 become trivial equalities. By contrast, M_- has an interesting, infinite, 2-periodic[1] R-free resolution

$$\cdots \xrightarrow{d_4} R(-7)^2 \xrightarrow{d_3} R(-5)^2 \xrightarrow{d_2} R(-3)^2 \xrightarrow{d_1} R(-1)^2 \xrightarrow{d_0} M_- \to 0,$$

in which $R(-d)$ denotes a free R-module of rank 1 having a basis element of degree d. Here the differential d_0 maps the two basis elements of $R(-1)^2$ onto x, y in M_-, while the differentials d_i for $i \geq 1$ can be chosen as follows:

$$d_i = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \text{ for even } i, \qquad d_i = \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} \text{ for odd } i.$$

[1] In fact, whenever R is a *hypersurface*, i.e.,

$$R = k[V]^G \cong k[f_1, \ldots, f_n, f_{n+1}]/(h)$$

for a single homogeneous relation h among the f_i's, there will always be such an R-free resolution of M which is eventually 2-periodic (see e.g., [31, §6] or [10]).

For any $m \geq 0$, this gives the following strict inequalities in $\mathbf{R}(\mathbb{C}(\Gamma)) \cong \mathbb{Z}$

$$\left[\sum_{i=0}^{m}(-1)^i \mathrm{Tor}_i^R(M_-, k)\right] = \begin{array}{l} [\mathbb{C}^2] = 2[\mathbf{1}] > [\mathbf{1}] = [U_-] \text{ if } m \text{ is even,} \\ [0] \;\; = 0[\mathbf{1}] < [\mathbf{1}] = [U_-] \text{ if } m \text{ is odd,} \end{array}$$

as predicted by Theorem 1.1.1 (i, ii). In the limit as $m \to \infty$, one makes sense of this by noting that $\mathrm{Tor}_i^R(M_-, \mathbb{C})_j$ vanishes unless the internal degree j and homological degree i satisfy $j = 2i + 1$. Hence one can calculate in $\mathbf{R}(\mathbb{C}(\Gamma))[[t]] \cong \mathbb{Z}[[t]]$ that

$$\begin{aligned} \sum_{i \geq 0}(-1)^i \; \left[\mathrm{Tor}_i^R(M_-, \mathbb{C})\right](t) &= 2[\mathbf{1}]t^1 - 2[\mathbf{1}]t^3 + 2[\mathbf{1}]t^5 - 2[\mathbf{1}]t^7 + \cdots \\ &= \frac{2t}{1+t^2}[\mathbf{1}]. \end{aligned}$$

This is a rational function of t with $t = 1$ as a regular value, and upon substituting $t = 1$, one obtains in $\mathbf{R}(\mathbb{C}(\Gamma))$

$$\sum_{i \geq 0}(-1)^i \left[\mathrm{Tor}_i^R(M_-, \mathbb{C})\right](t)\Bigg|_{t=1} = \frac{2t}{1+t^2}[\mathbf{1}]\Bigg|_{t=1} = \frac{2}{2}[\mathbf{1}] = [\mathbf{1}] = [U_-]$$

as predicted by Theorem 1.1.1(iii).

1.2. **Springer-Type Results.** Let k be a field, V a finite-dimensional k-vector space, G a finite subgroup of $GL(V)$. Suppose that $k[V]^G$ is a polynomial algebra, and c is a regular element of G. As with the Chevalley, Shephard-Todd, Mitchell Theorem, we wish to deduce a more general version of Springer's Theorem that applies to an arbitrary $(k(\Gamma), k(G))$-bimodule U for any finite group Γ. The $k(\Gamma)$-module of relative invariants $M := (U \otimes_k k[V])^G$ carries a commuting C-action in which C acts on $k[V]$ by scalars as before, and does nothing to the factor of U in $U \otimes_k k[V]$. In this way M becomes a graded $(k(\Gamma), k(C))$-bimodule. It is also possible to view U itself as a $(k(\Gamma), k(C))$-bimodule in a different way, namely with the action of C coming as the restriction of the action of G, of which C is a subgroup. From the theorem of Springer, it is not hard to deduce that if $k[V]^G$ is a polynomial algebra and both $|G|, |\Gamma|$ lie in $k^\times$, then as $(k(\Gamma), k(C))$-bimodules,

$$M \otimes_{k[V]^G} k \cong U.$$

Our next main result shows how to remove the hypothesis that $|G|, |\Gamma|$ lie in $k^\times$.

Theorem 1.2.1. *Let k be any field, V a finite-dimensional k-vector space of k and G a finite subgroup of $GL(V)$ with $k[V]^G$ a polynomial algebra. Let $C = \langle c \rangle$ be the cyclic subgroup generated by a regular element c in G with regular eigenvalue ω and U a finite-dimensional $(k(\Gamma), k(G))$-bimodule for some finite group Γ. Regard $M := (U \otimes_k k[V])^G$ as a $k[V]^G$-module and as a $(k(\Gamma), k(C))$-bimodule, where C acts on $k[V]$ via scalar multiplication* *determined by* $c^j(x_i) = \omega^j x_i$ *for* $i = 1, 2, \ldots, n$ *and* $x_1, \ldots, x_n$ *is a basis for* V^**.*

Then

$$\sum_{i \geq 0} (-1)^i \left[\mathrm{Tor}_i^{k[V]^G}(M, k) \right] = [U].$$

in $\mathbf{R}(k(\Gamma \times C))$*; the sum being finite since* $k[V]^G$ *is a polynomial algebra.*

Proving Theorem 1.2.1 (in particular, with ***no*** hypothesis on the field k) was one of our original motivations. It easily implies our main application, Corollary 1.2.2 below, which resolves in the affirmative both Conjecture 3 and Question 4 in [20].

Corollary 1.2.2. *Let* k *be a field,* V *a finite-dimensional* k*-vector space over* k *and* $H \subset G$ *two nested finite subgroups of* $GL(V)$*. Assume* $k[V]^G$ *is a polynomial algebra, and let* $C = \langle c \rangle$ *be the cyclic subgroup generated by a regular element* c *in* G*, with eigenvalue* ω *on some regular vector in* V *and* $\hat{\omega} \in \mathbb{C}$ *a complex lift of* ω*. Let* $\Gamma := N_G(H)/H$*, where* $N_G(H)$ *denotes the normalizer of* H *in* G*. Give* $k[V]^H$ *the structure of a* $(k(\Gamma), k(C))$*-bimodule in which* c *scales the variables* $x_1, \ldots, x_n$ *in* V^* *by* ω*, and* Γ *acts by linear substitutions. Let* $k(H \backslash G)$ *be the* k*-vector space with basis the right costs of* H *in* G *regarded as a* $(k(\Gamma), k(C))$*-bimodule where*

$$\gamma \cdot Hg \cdot c := \gamma Hgc = H\gamma gc.$$

Then in $\mathbf{R}(k(\Gamma \times C))$ *one has the equality*

$$\sum_{i \geq 0} (-1)^i \left[\mathrm{Tor}_i^{k[V]^G}(k[V]^H, k) \right] = [k(H \backslash G)] .$$

Ignoring the Γ*-action, this implies that the quotient of Hilbert series*

$$X(t) := \frac{[k[V]^H](t)}{[k[V]^G](t)},$$

is a polynomial in t*, and together with the* C*-action on the set* $X = H \backslash G$*, gives a triple* $(X, X(t), C)$ *that exhibits the* **cyclic sieving phenomenon** *of* [21]*: Namely, for each element* c^j *in* C *the cardinality of the fixed point set* $X^{c^j} \subset X$ *is given by evaluating* $X(t)$ *at the complex root-of-unity* $\hat{\omega}^j$ *of the same multiplicative order as* c^j*. In other words,*

$$|X^{c^j}| = [X(t)]_{t = \hat{\omega}^j} .$$

1.3. **A Further Generalization.** Theorem 1.2.1 follows from the more general Theorem 1.3.1, which does not make the assumption that $k[V]^G$ is polynomial and has other applications. The guiding principle in this generalization is to replace the condition that $k[V]^G$ be a polynomial algebra with the assumption it contains a Noether normalization $R \subseteq k[V]^G$ fulfilling certain key technical conditions. Recall that a **Noether normalization** for $k[V]^G$ is a polynomial subalgebra $R \subset k[V]^G$ over which $k[V]^G$ is a finitely generated module. To state our result requires some preliminaries.

Let V be a $(k(G), k(C))$-bimodule with G and C finite groups, so that C acts on $k[V]$ and $k[V]^G$. Suppose there exists a homogeneous and C-stable Noether normalization $R \subset k[V]^G \subset k[V]$, with the added property that the fiber $\Phi_v := \phi^{-1}(\phi(v))$ over the point $\phi(v)$ in the ramified covering $V \to \operatorname{Spec}(R)$ has free (but not necessarily transitive) G-action, and is stable under C. This means that C preserves the tower of inclusions

$$\mathfrak{m}_{\phi(v)} \subset R \subset k[V]^G \subset k[V],$$

where $\mathfrak{m}_{\phi(v)}$ is the (generally inhomogeneous) maximal ideal in R corresponding to $\phi(v)$ in $\operatorname{Spec}(R)$. The quotient ring $k[V]/\mathfrak{m}_{\phi(v)}k[V] =: A(\Phi_v)$ can be thought of as the coordinate ring for the fiber Φ_v regarded as a (possibly non-reduced) subscheme of the affine space V. This ring $A(\Phi_v)$ carries an interesting $(k(G), k(C))$-bimodule structure, whose precise description we defer until §3.5 where the extra generality is exploited.

However, it is worth mentioning here what this bimodule structure looks like under the hypotheses of Theorem 1.2.1, that is, if $k[V]^G$ is polynomial, so that we can choose $R = k[V]^G$, and where c is a regular element of G with eigenvalue ω on a regular eigenvector v. In this special case, if we take for C the group of scalar matrices in $GL(V)$ generated by ω times the identity matrix, then $A(\Phi_v) \cong k(G)$ carries the same $(k(G), k(C))$-bimodule structure as was described on $k(G)$ in Springer's theorem (Theorem 1.2.1).

Back in the general setting, given a $(k(\Gamma), k(G))$-bimodule U, one lets C act trivially on U and Γ act trivially on $k[V]$. In this way, the relative invariants $M := (U \otimes_k k[V])^G$ carry the structure of a graded $(k(\Gamma), k(C))$-bimodule, compatible with its R-module structure. Similarly $(U \otimes_k A(\Phi_v))^G$ carries the structure of a $(k(\Gamma), k(C))$-bimodule in which Γ acts only on the U factor, and C acts only on the $A(\Phi_v)$ factor.

Theorem 1.3.1. *Let k be a field, G, Γ, and C finite groups, and V a finite dimensional $(k(G), k(C))$-bimodule on which G acts faithfully (so $G \subset GL(V)$). Regard $V, k[V]$ and $k[V]^G$ as trivial $k(\Gamma)$-modules. Suppose there is a Noether normalization $R \subset k[V]^G$ that is stable under the action of C on $k[V]$. Suppose further that there is a vector v in V such that the fiber $\Phi_v := \phi^{-1}(\phi(v))$ containing v for the map $\phi : V \to \operatorname{Spec}(R)$ both carries a free (but not necessarily transitive) G-action and is stable under C. Denote by $\mathfrak{m}_{\phi(v)}$ the maximal ideal in R corresponding to $\phi(v)$ in $\operatorname{Spec}(R)$ and set $A(\Phi_v) = k[V]/\mathfrak{m}_{\phi(v)}k[V]$ which is a $k(C)$-module. Let U be a finite-dimensional $(k(\Gamma), k(G))$-bimodule regarded as a trivial $k(C)$-module.*

Then the relative invariants $M := (U \otimes_k k[V])^G$ satisfy

$$\sum_{i \geq 0} (-1)^i \ \left[\operatorname{Tor}_i^R(M, k)\right] = \left[(U \otimes_k A(\Phi_v))^G\right]$$

in $\mathbf{R}(k(\Gamma \times C))$; the sum being finite since R is a polynomial algebra.

From this we easily deduce Theorem 1.2.1 in Section 3.5. Section 4 illustrates a different application of Theorem 1.3.1, to character values of the binary icosahedral group.

2. Generalities for Rings, Modules, and Tor

We record here some issues surrounding $\mathrm{Tor}^R(M,k)$ and the action of a group on R-resolutions of M. Without the group action most of these results can be found in [27]. Because the group varies in the applications we denote it by the new symbol $\mathcal{G}$. It will always be assumed throughout Section 2 that

(i) $R = \oplus_{d\geq 0} R_d$ is a commutative, $\mathbb{N}$-graded, connected ($R_0 = k$), Noetherian k-algebra (i.e., finitely generated as an algebra over k) and

(ii) $\mathcal{G}$ is a group which acts on R by graded k-algebra automorphisms.

In addition we will consider finitely generated R-modules with a compatible homogeneous action of $\mathcal{G}$ in a sense conveniently described in terms of the **skew group algebra** $R \rtimes \mathcal{G}$ (see e.g., [2]). This is the free R-module with elements $\{t_g\}_{g\in\mathcal{G}}$ indexed by $\mathcal{G}$ as a basis, and whose multiplication is determined by the rule $rt_g \cdot st_h = r \cdot g(s)t_{gh}$ and bilinearity, for all $r, s \in R$ and $g, h \in \mathcal{G}$. We put a grading on $R \rtimes \mathcal{G}$ by requiring that an element rt_g has the same degree as r. An $R \rtimes \mathcal{G}$-module is the same thing as an R-module M with an action of $\mathcal{G}$ on M regarded as a graded abelian group by grading preserving group endomorphisms satisfying $g(rm) = g(r)g(m)$ and $g_1(g_2m) = (g_1g_2)m$. This is what we mean by a **compatible** action of $\mathcal{G}$. So the third standing assumption in this section is that

(iii) $M = \oplus_{d\geq 0} M_d$ is an $\mathbb{N}$-graded $R \rtimes \mathcal{G}$-module which is Noetherian (i.e., finitely generated) as an R-module.

The kind of graded $R \rtimes \mathcal{G}$-modules we will consider arises, for example in the situation discussed in the introduction: We have a $(k(G), k(C))$-bimodule V, so that C acts on $k[V], k[V]^G$ and possibly also on a Noether normalization $R \subset k[V]^G$, as well as compatibly on the R-module $M := (U \otimes_k k[V])^G$ for any $(k(\Gamma), k(G))$-bimodule U. In fact putting $\mathcal{G} = \Gamma \times C$, M becomes an $R \rtimes \mathcal{G}$-module.

2.1. **Review of Graded Resolutions.** Recall that, ignoring group actions, there always exist *graded R-free resolutions* $\mathcal{F}$ of M in which all terms are finitely generated, that is, an exact sequence

$$\cdots \overset{d_{i+1}}{\to} F_i \overset{d_i}{\to} F_{i-1} \overset{d_{i-1}}{\to} \cdots F_1 \overset{d_1}{\to} F_0 \overset{d_0}{\to} M \to 0$$

with each F_i a graded free R-module of finite rank, and grade-preserving differentials d_i. From any such resolution one can compute the bigraded k-vector space $\mathrm{Tor}^R(M,N) = \{\mathrm{Tor}_i^R(M,N)_j\}$ for any graded R-module N, by taking the homology of the tensored complex $\mathcal{F} \otimes_R N$. For $i \in \mathbb{N}$ $\mathrm{Tor}_i^R(M,N)$ is a graded k-vector space, the index i is called the **homological grading**, and the grading on $\mathrm{Tor}_i^R(M,N)$, namely the index j, is called the **internal** grading. Depending on the context we will use the notation $\mathrm{Tor}^R(M,N)$ for the bigraded Tor-functor, or its ungraded analog obtained by taking the direct sum of the homogeneous components $\{\mathrm{Tor}_i^R(M,N)_j\}$.

It is possible to choose the resolution $\mathcal{F}$ to be **minimal** in the sense that the ranks β_i of the resolvents $F_i \cong R^{\beta_i}$ are simultaneously all minimized; this turns out to be equivalent to each differential d_i having entries in $R_+ = \oplus_{i>0} R_i$. In particular, when $N = k = R/R_+$ is the trivial R-module, if the complex is minimal then $\mathcal{F} \otimes_R N$ becomes a complex of k-vector spaces with all zero differentials, showing that

$$\beta_i = \dim_k \mathrm{Tor}_i^R(M,k).$$

The length of a minimal resolution is called the **homological dimension** $\mathrm{hd}_R(M)$, that is, $\mathrm{hd}_R(M) := \min\{i : \mathrm{Tor}_i^R(M,k) \neq 0\}$. Note that $\mathrm{hd}_R(M)$ need not be finite. However, *Hilbert's syzygy theorem* asserts that when R is a polynomial algebra on n generators, one always has $\mathrm{hd}_R(M) \leq n$.

Given an $\mathbb{N}$-graded k-vector space $U = \sum_{d \geq 0} U_d$, let

$$\mathrm{start}(U) := \min\{d : U_d \neq 0\}.$$

The usual construction of a minimal free R-resolution $\mathcal{F}$ of M shows that it enjoys the property $\mathrm{start}(F_{i+1}) > \mathrm{start}(F_i)$. We will show in Section 2.3 that a similar property holds after incorporating a finite group action.

2.2. **The Group Action on** Tor**.** We start by pointing out that for $R \rtimes \mathcal{G}$-modules M and N (where $\mathcal{G}$ is a group which acts on R by graded k-algebra automorphisms) there are diagonal actions of R and $\mathcal{G}$ on $M \otimes_R N$ and also on $\mathrm{Tor}_i^R(M,N)$ making them into $R \rtimes \mathcal{G}$-modules. Since R is commutative, we allow ourselves to take the tensor product of two left R-modules. We claim that for each $g \in \mathcal{G}$ the map $M \times N \to M \otimes_R N$ given by $(m,n) \mapsto g(m) \otimes g(n)$ is R-balanced. To establish this we must show that for each $r \in R$ we have $g(rm) \otimes g(n) = g(m) \otimes g(rn)$. This is so because

$$g(rm) \otimes g(n) = g(r)g(m) \otimes g(n) = g(m) \otimes g(r)g(n) = g(m) \otimes g(rn).$$

From this we obtain the diagonal action of $R \rtimes \mathcal{G}$ on $M \otimes_R N$.

In fact for each $g \in \mathcal{G}$ we have a natural transformation from the functor

$$- \otimes_R N : R \rtimes \mathcal{G}\text{-mod} \to R\text{-mod}$$

to itself, giving a functor

$$- \otimes_R N : R \rtimes \mathcal{G}\text{-mod} \to R \rtimes \mathcal{G}\text{-mod}.$$

We next show that the diagonal action of $\mathcal{G}$ extends to an action on $\mathrm{Tor}_i^R(M,N)$ for $i > 0$. Regard $\mathrm{Tor}_i^R(-,N)$ as a functor $R \rtimes \mathcal{G}$-mod $\to$ R-mod. For each $g \in \mathcal{G}$ and $i \in \mathbb{N}$ we construct a natural transformation $\eta_{g,i}$ from $\mathrm{Tor}_i^R(-,N)$ to itself so that these maps also commute with the boundary maps in the long exact sequences which arise from any short exact sequence of $R \rtimes \mathcal{G}$-modules $0 \to M_1 \to M_2 \to M_3 \to 0$. For $i = 0$ the natural transformation will be the one already constructed. To extend this for arbitrary i, we may take any complex of projective $R \rtimes \mathcal{G}$-modules $\mathcal{P} = (\cdots \to P_2 \to P_1 \to P_0 \to 0)$ which is acyclic except in degree zero where its homology is M. Since $R \rtimes \mathcal{G}$ is free as an R-module this is also a projective resolution of M as an R-module, so $\mathrm{Tor}_i^R(M,N) = H_i(\mathcal{P} \otimes_R N)$. As the action of $\mathcal{G}$ is by natural transformations of the functor $- \otimes_R N$ it passes to an action on the complex $\mathcal{P} \otimes_R N$ and hence to an action on its homology. The verification that this action commutes with the boundary homomorphisms is a standard argument in homological algebra.

We finally observe that if higher natural transformations exist with these properties, they must be unique. This follows from a homological degree shifting argument (cf [6, Chapter III]), since given any $R \rtimes \mathcal{G}$-module M one may take a short exact sequence $0 \to K \to P \to M \to 0$ where P is a projective $R \rtimes \mathcal{G}$-module. This gives a long exact sequence

$$0 = \mathrm{Tor}_i^R(P,N) \to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_{i-1}^R(K,N) \to \mathrm{Tor}_{i-1}^R(P,N) \to \cdots$$

and the specification of $\eta_{g,i-1}$ on $\mathrm{Tor}_{i-1}^R(K,N)$ and $\mathrm{Tor}_{i-1}^R(P,N)$ (which is only non-zero for $i = 1$) determine the specification of $\eta_{g,i}$ on $\mathrm{Tor}_i^R(M,N)$ since these maps commute with the connecting homomorphisms. Then the uniqueness implies that $\mathrm{Tor}_i^R(M,N)$ becomes an $R \rtimes \mathcal{G}$-module, since the relations on the action of $\mathcal{G}$ which are needed for this hold if $i = 0$, hence also for higher values of i.

2.3. **The Group Action on Resolutions.** Given these preliminaries we can state and prove our first rationality result. This proposition can be regarded as an equivariant generalization of the theorem of Hilbert–Serre (see e.g., [27, Theorem 4.2]) on the rationality of Poincaré series of graded Noetherian modules over Noetherian k-algebras.

Proposition 2.3.1. *Let R be a commutative $\mathbb{N}$-graded Noetherian k algebra, $\mathcal{G}$ is a group which acts on R by graded k-algebra automorphisms, and M a Noetherian $R \rtimes \mathcal{G}$-module.*

(i) There exists an R-resolution $\mathcal{F} = \{F_i\}_{i \geq 0}$ of M in which each resolvent F_i is not only a free R-module but also a $k(\mathcal{G})$-module, the maps are $k(\mathcal{G})$-module morphisms, and $\mathrm{start}(F_{i+1}) > \mathrm{start}(F_i)$. *Consequently, the infinite sum*

$$\sum_{i \geq 0} (-1)^i [\mathrm{Tor}_i^R(M,k)](t)$$

gives rise to a well-defined element in $\mathbf{R}(k(\mathcal{G}))[[t]]$.

(ii) When $k(\mathcal{G})$ is semisimple (i.e., $|\mathcal{G}| \in k^\times$), the resolution in (i) can in addition be chosen minimal as an R-resolution.

(iii) If $\mathrm{hd}_R(M)$ is finite, the resolution in (i) or in (ii) can in addition be chosen with length $\mathrm{hd}_R(M)$.

(iv) In $\mathbf{R}(k(\mathcal{G}))[[t]]$, the series $[R](t)$ is invertible, and one has the following relation:

$$\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^R(M,k)](t) = \frac{[M](t)}{[R](t)}.$$

(v) All three series

$$[R](t)\,,[M](t)\,,\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^R(M,k)](t)$$

in $\mathbf{R}(k(\mathcal{G}))[[t]]$ actually lie in the subring $\mathbb{Q}(t)\otimes_{\mathbb{Z}}\mathbf{R}(k(\mathcal{G}))$. That is, for each simple $k(\mathcal{G})$-module S, the power series which is the coefficient of $[S]$ actually lies in $\mathbb{Q}(t)$.

(vi) For each simple $k(\mathcal{G})$-module S, the coefficient series of $[S]$ in $[M](t)$ and $[R](t)$ have poles at $t = 1$ of order at most the Krull dimension of R.

Proof. (i): Because M is finitely-generated as an R-module, the quotient M/R_+M is a finite-dimensional, graded k-vector space. Any homogeneous k-basis $\{\bar{m}_j\}$ for M/R_+M lifts to a minimal homogeneous generating set $\{m_j\}$ for M as an R-module, by the graded version of Nakayama's Lemma. Let U be the $k(\mathcal{G})$-submodule of M generated by any choice of such lifts $\{m_j\}$. Then U is a graded finite-dimensional subspace of M because $\mathcal{G}$ is finite and acts in a degree-preserving fashion on M. Start a resolution $\mathcal{F}$ with the surjection

$$F_0 := R\otimes_k U \xrightarrow{d_0} M\,, \quad \text{where} \quad r\otimes u \longmapsto ru.$$

Observe that the diagonal action of $k(\mathcal{G})$ on $F_0 := R\otimes_k U$ is required here, both to make d_0 a $k(\mathcal{G})$-module morphism, and to make the R-module structure on F_0 compatible with the $k(\mathcal{G})$-module structure of R. Observe also that $U \cong k\otimes_R F_0$, a relationship which will be used in proving (iv).

Replacing M by $\ker(d_0)$, we can iterate this process, and produce the desired resolution $\mathcal{F}$, provided we can show that the inequality

$$\mathrm{start}(\ker(d_0)) > \mu := \mathrm{start}(M) = \mathrm{start}(F_0)$$

holds. However, this follows easily from the observation that the restriction of the above map d_0 to the μ^{th} homogeneous components is the k-vector space isomorphism

$$R_0\otimes_k U_\mu = k\otimes_k U_\mu \xrightarrow{d_0} U_\mu = M_\mu\,, \quad \text{where} \quad 1\otimes u \longmapsto u\,,$$

and hence $\ker(d_0)$ is nonzero only in degrees strictly larger than μ.

(ii): If $k(\mathcal{G})$ is semisimple, then in the construction of $\mathcal{F}$ in (i), the $k(\mathcal{G})$-submodule $U \subset M$ is a direct summand so there is a $k(\mathcal{G})$-module direct sum decomposition $M = U \oplus R_+M$. Since $U \cong M/R_+M$, iterating this construction will produce a minimal resolution.

(iii): This we prove by induction on $\mathrm{hd}_R(M)$. In the base case, i.e., where $\mathrm{hd}_R(M) = 0$, M is a free R-module and the assertion is trivial. In the inductive step, note that after one step of the construction in (i) or (ii), there is a short exact sequence

$$0 \to \ker(d_0) \to F_0 \to M \to 0$$

whose long exact sequence in $\mathrm{Tor}^R(-,k)$ shows that

$$\mathrm{hd}_R(\ker(d_0)) = \mathrm{hd}_R(M) - 1.$$

Applying the inductive hypothesis to $\ker(d_0)$ gives the result.

(iv): The fact that $[R](t)$ lies in $\mathbf{R}(k(\mathcal{G}))[[t]]^\times$ follows from the assumption that R is a connected k-algebra with $\mathcal{G}$ acting trivially on $R_0 = k$. This means that the series expansion of $[R](t)$ begins $[\mathbf{1}] + [R_1]t^1 + [R_2]t^2 + \cdots$, and $[\mathbf{1}]$ is a unit of $\mathbf{R}(k(\mathcal{G}))$.

For the remaining assertion, start with a resolution $\mathcal{F}$ of M produced as in (i). There is the following string of equalities in $\mathbf{R}(k(\mathcal{G}))[[t]]$, which are justified below.

$$\begin{aligned}
[M](t) &= \sum_{i\geq 0} (-1)^i [F_i](t) \\
&= \sum_{i\geq 0} (-1)^i [R \otimes_k (k \otimes_R F_i)](t) \\
&= \sum_{i\geq 0} (-1)^i [R](t) \cdot [k \otimes_R F_i](t) \\
&= [R](t) \sum_{i\geq 0} (-1)^i \cdot [k \otimes_R F_i](t) \\
&= [R](t) \sum_{i\geq 0} (-1)^i \cdot [\mathrm{Tor}_i^R(M,k)](t)
\end{aligned}$$

The first equality comes from looking at the Euler characteristic for the (finite) exact sequence in each homogeneous component.

The second equality comes from the fact that F_i is constructed as $R \otimes_k U_i$ and $U_i \cong k \otimes_R F_i$, so $F_i \cong R \otimes_k (k \otimes_R F_i)$.

The third equality comes from the fact that in the isomorphism $F_i \cong R \otimes_k (k \otimes_R F_i)$ the action of $\mathcal{G}$ on the tensor product on the right is diagonal, and this tensor product defines the product in $\mathbf{R}(k(\mathcal{G}))[[t]]$.

The fourth equality is trivial.

The fifth equality holds because $\mathrm{Tor}^R(M,k)$ is the homology of the complex $\mathcal{F} \otimes_R k$. In each homogeneous component $(\mathcal{F} \otimes_R k)_d$ one has a finite complex of finite-dimensional k-vector spaces, and the alternating sum $\mathcal{F}_i \otimes_R k$ over i represents the same element in $\mathbf{R}(k(\mathcal{G}))$ as the alternating sum of the $\mathrm{Tor}_i^R(M,k)_d$ in that component (i.e., taking homology preserves Euler characteristics).

(v): It suffices to prove the assertion for $[M](t)$; one then takes $M = R$ to deduce it for $[R](t)$, and uses (iv) to deduce it for $\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^R(M,k)](t)$.

To prove $[M](t)$ is invertible, one can reduce to the case where R is a polynomial algebra $A = k[f_1,\ldots,f_n]$ with trivial $\mathcal{G}$-action as follows. Note that since R is Noetherian, by a result of Emmy Noether $R^{\mathcal{G}}$ is also Noetherian. Hence by the Noether Normalization Lemma, $R^{\mathcal{G}}$ contains a homogeneous system of parameters $f_1,\ldots,f_n$, and we put $A = k[f_1,\ldots,f_n]$. The ring extensions $A \hookrightarrow R^{\mathcal{G}} \hookrightarrow R$ are both integral (i.e. module-finite), and hence M is also a finitely generated A-module.

In this case, one can apply (iv) to give

$$[M](t) = \left[\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^R(M,k)](t)\right]\cdot[A](t). \tag{2.3.1}$$

Note that the sum on the right is finite, i.,e., lies in $\mathbf{R}(k(\mathcal{G}))[t]$, because Hilbert's syzygy theorem says $\mathrm{Tor}^A(M,k)$ is finite dimensional. Note also that because $\mathcal{G}$ acts trivially on A, one has

$$[A](t) = \left(\prod_{i=1}^{n}\frac{1}{1-t^{\deg(f_i)}}\right)[\mathbf{1}],$$

which is an element of $\mathbb{Q}(t)$ times the class $[\mathbf{1}]$ of the trivial module. Hence $[M](t)$ has $\mathbb{Q}(t)$ coefficients.

(vi): Again it suffices to prove it for $[M](t)$, and then take $M = R$ to deduce it for $[R](t)$. For $[M](t)$ it is implied by equation (2.3.1) and the comments after it, as the sum $\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^R(M,k)](t)$ has $\mathbb{Z}[t]$ coefficients and the pole at $t = 1$ in $[A](t)$ is the Krull dimension of A. This is the same as the Krull dimension of R since $A \hookrightarrow R$ is an integral extension. ☐

2.4. **A Short Review of Brauer theory.** At several points we will need facts about the Grothendieck ring $\mathbf{R}(k(\Gamma))$ for a finite group Γ which can be conveniently deduced from the theory of Brauer characters. We review this theory here and refer to [23, Part III] for details.

Let p denote the characteristic of the ground field k. We say that an element γ in Γ is **p-regular** if its order lies in $k^\times$. Let m be the least common multiple of the orders of all p-regular elements of Γ, and let ζ be a primitive m^{th} root of unity in some extension field of k. Then every p-regular element γ in Γ acting on a finite-dimensional $k(G)$-module U has all its eigenvalues among the group of m^{th} roots of unity $\mu_m(k(\zeta)^\times) = \langle\zeta\rangle$. Pick a primitive complex m^{th} root of unity $\hat{\zeta} \in \mathbb{C}$ which will lift ζ, and consider the resulting homomorphism which lifts m^{th} roots of unity from $k(\zeta)$ to $\mathbb{C}$:

$$\begin{aligned} \mu_m(k(\zeta)^\times) &\xrightarrow{\text{lift}} \mu_m(\mathbb{C}^\times) \\ \zeta^j &\longmapsto \hat{\zeta}^j. \end{aligned}$$

The **Brauer character value** $\chi_U(\gamma)$ for γ is then defined to be the sum of the lifts of the eigenvalues of g on U. Furthermore the field $k(\zeta)$ is a splitting field for Γ by a theorem of Brauer (see e.g., [23, §12.3, Theorem 24]).

The Brauer character χ_U of a $k(\Gamma)$-module U determines the composition factors of U, and this has several important consequences. To begin with, the collection of restriction homomorphisms

$$\mathbf{R}(k(\Gamma)) \longrightarrow \mathbf{R}(k\langle\gamma\rangle),$$

where γ ranges over all p-regular elements γ, determines elements of $\mathbf{R}(k(\Gamma))$ uniquely; that is, the map $\mathbf{R}(k(\Gamma)) \to \bigoplus_\gamma \mathbf{R}(k\langle\gamma\rangle)$ is injective. It implies also that whenever one has a field extension $k \hookrightarrow K$, the map $\mathbf{R}(k(\Gamma)) \stackrel{\psi_{k,K}}{\to} \mathbf{R}(K(\Gamma))$ that is induced by extension of scalars $U \mapsto K \otimes_k U$ is injective, since the Brauer character of a module remains the same after extending scalars. So to prove an equality in $\mathbf{R}(k(\Gamma))$ it suffices to prove the equality in $\mathbf{R}(k\langle\gamma\rangle)$ for the p-regular elements γ in Γ.

If $\gamma \in \Gamma$ is a p-regular element then $k\langle\gamma\rangle$ is semisimple. Over a splitting field $k(\zeta)$ the simple $k\langle\gamma\rangle$-modules U_j are all 1-dimensional and are indexed by $j \in \mathbb{Z}/d\mathbb{Z}$, where d is the order of γ, with γ acting as the scalar ω^j for some primitive d^{th} root of unity ω in $k^\times$. An element in $\mathbf{R}(k\langle\gamma\rangle)$ is determined by the (virtual) composition multiplicities of each U_j. If this element is of the form $[U]$ for some genuine $k(\Gamma)$-module U, then $[U]$ will be determined by the dimensions $\dim_k(U \otimes_k U_j)^{\langle\gamma\rangle}$ of its U_j-isotypic components.

Observe also that if one has a commuting action of another finite group Γ', and one wants to prove an equality in $\mathbf{R}(k(\Gamma\times\Gamma'))$, it suffices to prove it in $\mathbf{R}((k\langle\gamma\rangle\times\Gamma'))$ for each p-regular $\gamma \in \Gamma$. Furthermore, this can be done for genuine $k(\Gamma\times\Gamma')$-modules by proving equality in $\mathbf{R}(k(\Gamma'))$ for each isotypic component.

2.5. **The Case of Domains with Trivial Group Action.** We assume the notations introduced in Proposition 2.3.1, and in addition require that the graded k-algebra R be an integral domain on which Γ-acts trivially. We let K be the fraction field of R and recall from the discussion of Section 2.4 that extension of scalars gives an inclusion of Grothendieck rings $\psi_{k,K}$: $\mathbf{R}(k(\Gamma)) \to \mathbf{R}(K(\Gamma))$. We will make various assertions about isomorphism of $K(\Gamma)$-modules, but where these modules are in fact defined over k we may also deduce a corresponding result for $k(\Gamma)$ modules which we leave to the reader to formulate.

The main result in this section is an abstract version of Theorem 1.1.1, from which Theorem 1.1.1 will immediately follow. Before we state it, we present a lemma which will be needed in the proof.

Lemma 2.5.1. *Let M be a finitely generated graded $R(\Gamma)$-module, where R is a commutative graded Noetherian k-algebra on which Γ acts trivially. Assume that R is an integral domain and let $R' \subseteq R$ be a graded subring* *over which R is integral. Let the fields of fractions of R and R' be K and K', respectively. Then the map*

$$\varphi : K' \times M \longrightarrow K \otimes_R M$$

sending $(a, m) \in K' \times M$ to $a \otimes_R m \in K \otimes_R M$ induces an isomorphism of $K'(\Gamma)$-modules

$$K' \otimes_{R'} M \to K \otimes_R M.$$

Note that here K is regarded as a (K', R)-bimodule, and may even be regarded as a $(K'(\Gamma), R(\Gamma))$-bimodule, on which Γ acts trivially.

Proof. Note that the map φ is R'-balanced simply because $R' \subset R$. Hence it induces a well-defined map $\varphi : K' \otimes_{R'} M \to K \otimes_R M$, which one can see is K'-linear and even a $K'(\Gamma)$-morphism.

It remains to show that it is a K'-vector space isomorphism, which is facilitated by first observing that the K'-dimensions of the domain and range are the same, viz.,

$$\begin{aligned} \dim_{K'} \big(K' \otimes_{R'} M\big) &= \operatorname{rank}_{R'}(M) \\ &= \operatorname{rank}_{R'}(R) \cdot \operatorname{rank}_R(M) \\ &= [K : K'] \cdot \dim_K (K \otimes_R M) \\ &= \dim_{K'} (K \otimes_R M) . \end{aligned}$$

Thus it suffices to show φ is surjective. For this one need only check for any $s, r \in R$ with $r \neq 0$, and any $m \in M$, that the decomposable tensor $\frac{s}{r} \otimes_R m$ is in the image of φ. For this we use that R is integral over R', so there is a dependence

$$r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$$

with $a_i \in R'$. We may assume $a_0 \neq 0$, since one can divide by r in the domain R. Let

$$m' := -(r^{n-1} + a_{n-1} r^{n-2} + \cdots a_2 r + a_1) m,$$

which is an element of M, satisfying

$$r \cdot m' = a_0 \cdot m.$$

Hence

$$\frac{s}{r} \otimes_R m = \frac{s}{r a_0} \otimes_R a_0 m = \frac{s}{r a_0} \otimes_R r m' = \frac{1}{a_0} \otimes_R s m' = \varphi \left(\frac{1}{a_0} \otimes_{R'} s m' \right)$$

lies in the image of φ. □

Theorem 2.5.2. *Let M be a finitely generated graded $R(\Gamma)$-module, where R is a commutative graded connected Noetherian k-algebra on which Γ acts trivially. Assume in addition that R is an integral domain.*

(i) In $\mathbf{R}(K(\Gamma))$ *we have*

$$[K \otimes_k (M \otimes_R k)] \geq [K \otimes_R M].$$

Furthermore, one has equality if and only if M *is* R*-free, in which case the* $K(\Gamma)$*-module* $K \otimes_R M$ *has a filtration*

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d = K \otimes_R M$$

so that for each j *there is an isomorphism of* $K(\Gamma)$*-modules*

$$K \otimes_k (M \otimes_R k)_j \to F_j / F_{j-1}$$

where $(M \otimes_R k)_j$ *denote the* j^{th} *homogeneous component of* $M \otimes_R k$*.*

(ii) More generally, for any $m \geq 0$*, in* $\mathbf{R}(k(\Gamma))$*,*

$$\sum_{i=0}^{m} (-1)^i \left[K \otimes_k \operatorname{Tor}_i^R(M,k)\right] \begin{cases} \geq [K \otimes_R M] & \text{if } m \text{ is even,} \\ \leq [K \otimes_R M] & \text{if } m \text{ is odd,} \end{cases}$$

and equality holds if and only if $\operatorname{hd}_R(M) \leq m$*, that is, if and only if* $\operatorname{Tor}_i^R(M,k)$ *vanishes for* $i > m$*.*

(iii) Even if $\operatorname{hd}_R(M)$ *is not finite, the formal power series*

$$\sum_{i \geq 0} (-1)^i \ \left[K \otimes_k \operatorname{Tor}_i^R(M,k)\right](t)$$

of $\mathbf{R}(K(\Gamma))[[t]]$ *is a rational function of* t *so lies in* $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \mathbf{R}(K(\Gamma))$*. Moreover,* $t = 1$ *is a regular value of this rational function and*

$$\sum_{i \geq 0} (-1)^i \ \left[K \otimes_k \operatorname{Tor}_i^R(M,k)\right](t) \Big|_{t=1} = [K \otimes_R M] \quad \text{ in } \mathbf{R}(K(\Gamma)).$$

Observe in the statement of this result that the action of R on K coming from the inclusion $R \subseteq K$ is not the same as the action coming from the composite homomorphism $R \to k \hookrightarrow K$, thus distinguishing $K \otimes_R M$ from $K \otimes_k (M \otimes_R k)$.

Proof. We begin by proving (i). Let $\{e_\alpha\}$ be a minimal homogeneous R-spanning subset for M. Then $\{e_\alpha \otimes 1\}$ forms a k-basis for $M \otimes_R k$, by the graded version of Nakayama's Lemma. Hence $\{1 \otimes (e_\alpha \otimes 1)\}$ forms a K-basis for $K \otimes_k (M \otimes_R k)$.

Also $\{1 \otimes e_\alpha\}$ is a K-spanning set for $K \otimes_R M$. Filter $K \otimes_R M$ by letting F_j be the K-span of $1 \otimes e_\alpha$ for which e_α has degree at most j. Then $K \otimes_k (M \otimes_R k)$ has a direct sum decomposition coming from its inherited grading, and there is a composite mapping

$$(K \otimes_k (M \otimes_R k))_j \to \quad F_j \quad \to F_j/F_{j-1}$$

defined by

$$1 \otimes (e_\alpha \otimes 1) \quad \mapsto 1 \otimes e_\alpha \mapsto \ \overline{1 \otimes e_\alpha}$$

where e_α is assumed to have degree exactly j. This composite mapping is a surjection of $K(\Gamma)$-modules.

These surjections show the inequality asserted in (i). One has equality if and only if all these surjections are isomorphisms, that is, if and only if the $\{1 \otimes e_\alpha\}$ are a K-basis for $K \otimes_R M$, which happens if and only if they are K-linearly independent. This in turn happens if and only if the $\{e_\alpha\}$ are R-linearly independent, and hence an R-basis for M, so that M is free over R.

We turn next to the proof of (ii) and (iii). In both of these proofs, it is convenient to reduce to the case where $|\Gamma|$ lies in $k^\times$ and hence $k(\Gamma)$ is semisimple: Recall from Section 2.4 that virtual modules in $\mathbf{R}(k(\Gamma))$ are determined by their restrictions to the cyclic subgroups generated by p-regular elements $\gamma \in \Gamma$, so one may replace Γ with $\langle\gamma\rangle$ without loss of generality.

For the proof of (ii), semisimplicity of $k(\Gamma)$ allows us to write down the first m steps in a minimal R-free resolution of M as in Proposition 2.3.1 (ii), so that all differentials are $k(\Gamma)$-module maps. Let L denote the kernel after the m^{th} stage, so that one has the exact sequence

$$0 \to L \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Applying the functor $K \otimes_R (-)$ is the same as a localization, and hence gives rise to an exact sequence, whose i^{th} term for $i = 0, 1, \ldots, m$ looks like

$$K \otimes_R F_i \;\cong\; K \otimes_R R \otimes_k F_i / R_+ F_i \;\cong\; K \otimes_k \operatorname{Tor}_i^R(M, k)$$

due to minimality of the resolution. This shows that

$$(-1)^m \left(\sum_{i=0}^{m} (-1)^i \; [K \otimes_k \operatorname{Tor}_i^R(M, k)] - [K \otimes_R M] \right) = [K \otimes_R L] \geq 0$$

in $K(\Gamma)$, which gives the inequality in $\mathbf{R}(K(\Gamma))$ asserted in (ii), with equality if and only if $K \otimes_R L = 0$. Since L is a submodule of the free R-module F_m, it is torsion-free as an R-module and hence $K \otimes_R L = 0$ if and only if $L = 0$. Minimality of the resolution then shows that vanishing of L (which is sometimes called the $(m+1)^{st}$ *syzygy module for* M) is equivalent to $\mathrm{hd}_R(M) \leq m$.

For the proof of (iii), we note by Hilbert's syzygy theorem that it holds when R is a polynomial algebra as a special case of (ii).

To deduce the general case of (iii), extend the field k if necessary in order to pick a graded Noether normalization $R' \subseteq R$, that is a graded polynomial subalgebra R' over which R is module-finite, and let $K' \subseteq K$ be the associated extension of fraction fields with degree $[K : K']$. We will take advantage of the injective ring homomorphisms from Section 2.4

$$\begin{array}{ccccc} \mathbf{R}(k(\Gamma)) & \overset{\psi_{k,K'}}{\hookrightarrow} & \mathbf{R}(K'(\Gamma)) & \overset{\psi_{K',K}}{\hookrightarrow} & \mathbf{R}(K'(\Gamma)) \\ \mathbf{R}(k(\Gamma))[[t]] & \overset{\psi_{k,K'}}{\hookrightarrow} & \mathbf{R}(K'(\Gamma))[[t]] & \overset{\psi_{K',K}}{\hookrightarrow} & \mathbf{R}(K'(\Gamma))[[t]]. \end{array}$$

that arise by extension of scalars in each case. Applying Proposition 2.3.1 (iv) twice, one has in $\mathbf{R}(k(\Gamma))[[t]]$ that

$$\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^R(M,k)](t) = \frac{[M](t)}{[R](t)} = \frac{[M](t)}{[R'](t)}\cdot\frac{[R'](t)}{[R](t)}$$
$$= \left(\sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^{R'}(M,k)](t)\right)\cdot\frac{[R'](t)}{[R](t)}.$$

Applying the map $\psi_{k,K'}$, one concludes that in $\mathbf{R}(K'(\Gamma))[[t]]$ one has

$$\sum_{i\geq 0}(-1)^i[K'\otimes_k \mathrm{Tor}_i^R(M,k)](t)$$
$$(2.5.1)\qquad = \left(\sum_{i\geq 0}(-1)^i[K'\otimes_k \mathrm{Tor}_i^{R'}(M,k)](t)\right)\cdot\psi_{k,K'}\left(\frac{[R'](t)}{[R](t)}\right).$$

The first factor on the right in (2.5.1) has $t=1$ as a regular value in $\mathbf{R}(K'(\Gamma))$, because R' is a polynomial algebra, and the case already proven shows that the value taken there is $[K'\otimes_{R'} M]$. For the second factor on the right in (2.5.1), note that both R', R carry trivial Γ-actions, and hence either [4, Lemma 2.4.1 (iii) on page 20] or [25, Prop. 5.5.2 on page 123] shows that this factor also has $t=1$ as a regular value in $\mathbf{R}(K'(\Gamma))$, taking the value $\frac{1}{[K:K']}[\mathbf{1}]$. Consequently, the left side of (2.5.1) has $t=1$ as a regular value in $\mathbf{R}(K'(\Gamma))$, taking the the value

$$\sum_{i\geq 0}(-1)^i[K'\otimes_k \mathrm{Tor}_i^R(M,k)](t)\Big|_{t=1} = \frac{1}{[K:K']}[K'\otimes_{R'} M]$$
$$= \frac{1}{[K:K']}[K\otimes_R M]$$

where the second equality uses Lemma 2.5.1. Applying the map $\psi_{K',K}$, one concludes that the element

$$\sum_{i\geq 0}(-1)^i[K\otimes_k \mathrm{Tor}_i^R(M,k)](t)$$

in $\mathbf{R}(K(\Gamma))[[t]]$ also has $t=1$ as a regular value in $\mathbf{R}(K(\Gamma))$, taking the value

$$\sum_{i\geq 0}(-1)^i[K\otimes_k \mathrm{Tor}_i^R(M,k)](t)\Big|_{t=1} = \frac{1}{[K:K']}\psi_{K',K}[K\otimes_R M]$$
$$= \frac{1}{[K:K']}[K\otimes_{K'} K\otimes_R M]$$
$$= [K\otimes_R M].$$

Here the last equality uses the fact that both K', K carry trivial Γ-actions, and $K\otimes_{K'} K\cong K^{[K:K']}$. □

Remark 2.5.3. Simple examples with $\mathrm{Tor}^R(M,k)$ infinite show that the assumption that Γ act trivially on R in Theorem 2.5.2 cannot in general be removed. For example, consider the following inclusions of algebras.

$$\underbrace{\mathbb{C}[x^2,y^2]}_{R'} \subset \underbrace{\mathbb{C}[x^2,xy,y^2]}_{R} \subset \underbrace{\mathbb{C}[x,y]}_{M}$$

Let $\Gamma = \mathbb{Z}/4\mathbb{Z}$ act compatibly on R', R, M via the scalar substitution given by $x, y \mapsto ix, iy$ (where $i^2 = -1$ in $\mathbb{C}$), so that their d^{th} homogeneous components are scaled by i^d. Then $\mathbf{R}(k(\Gamma)) \cong \mathbb{Z}[\alpha]/(\alpha^4-1)$, where α represents the class of the 1-dimensional $k(\Gamma)$-module that is scaled by i. In $\mathbf{R}(k(\Gamma))[[t]]$ there are equalities

$$\begin{aligned}
[\mathrm{Tor}^R(M,k)](t) &= \frac{[M](t)}{[R](t)} \quad (\text{ by Proposition 2.3.1(iv) above}) \\
&= \frac{[R'](t)[\mathrm{Tor}^{R'}(M,k)](t)}{[R'](t)[\mathrm{Tor}^{R'}(R,k)](t)} \quad (\text{ again by Proposition 2.3.1(iv) above}) \\
&= \frac{(1+\alpha t)^2}{1+\alpha^2 t^2} \\
&= 1 + \frac{2\alpha t}{1+\alpha^2t^2} \\
&= 1 + \alpha \cdot \frac{2t}{1-t^4} + \alpha^2 \cdot 0 + \alpha^3 \cdot \frac{-2t^3}{1-t^4}
\end{aligned}$$

and the last of these does not have $t = 1$ as a regular value. Note that here R is not fixed by Γ.

2.6. **Proof of Theorem 1.1.1.** To prove Theorem 1.1.1, one applies Theorem 2.5.2 in the special case where $R = k[V]^G$. Then K is its fraction field $k(V)^G$, and one has the R-module $M = (U \otimes_k k[V])^G$, with action by Γ coming from the fact that U is a $(k(\Gamma), k(G))$-bimodule. What remains to show in this situation is that

$$[K \otimes_R M] = [K \otimes_k U] \quad \text{in } \mathbf{R}(K(\Gamma)).$$

In fact, these two $K(\Gamma)$-modules are *isomorphic* as the following string of equalities and isomorphisms proves.

$$\begin{aligned}
K \otimes_R M &\overset{(1)}{=} k(V)^G \otimes_{k[V]^G} (U \otimes_k k[V])^G \\
&\overset{(2)}{\cong} (U \otimes_k k(V))^G \\
&\overset{(3)}{\cong} (U \otimes_k KG)^G \\
&\overset{(4)}{\cong} K \otimes_k (U \otimes_k k(G))^G \\
&\overset{(5)}{\cong} K \otimes_k U.
\end{aligned}$$

These may be justified as follows. Equality $\overset{(1)}{=}$ is just a definition. The isomorphism $\overset{(2)}{\cong}$ is given by the map

$$\begin{aligned} k(V)^G \otimes_{k[V]^G} (U \otimes_k k[V])^G \to & \quad (U \otimes_k k(V))^G \\ \frac{g}{h} \otimes \sum_i (u_i \otimes f_i) \longmapsto & \quad \sum_i u_i \otimes f_i \frac{g}{h}. \end{aligned}$$

whose inverse sends

$$\sum_i u_i \otimes \frac{g_i}{h_i} = \sum_i u_i \otimes \frac{\hat{g}_i}{h} \longmapsto \frac{1}{h} \otimes \left(\sum_i u_i \otimes \hat{g}_i \right)$$

where $\hat{g}_i, h$ are chosen so that $\frac{\hat{g}_i}{h} = \frac{g_i}{h_i}$ and h is G-invariant (e.g. choose h to be the product of the finitely many images $g(h_i)$ as i varies and as g varies through the finite group G). The isomorphism $\overset{(3)}{\cong}$ comes from the Normal Basis Theorem [15, Theorem VIII.13.1] applied to the Galois extension

$$K = k(V)^G \subset k(V),$$

which asserts $k(V) \cong K(G)$ as $k(G)$-modules. The isomorphism $\overset{(4)}{\cong}$ comes from the fact that G acts trivially on $K = k(V)^G$. The isomorphism $\overset{(5)}{\cong}$ comes from the k-linear map defined by

$$\begin{aligned} U \to & \quad (U \otimes_k k(G))^G \\ u \longmapsto & \quad \sum_{g \in G} u g^{-1} \otimes t_g \end{aligned}$$

where t_g is the k-basis element in $k(G)$ indexed by g.

This completes the proof of Theorem 1.1.1.

2.7. **Remarks on Group Cohomology and other related Constructions.** We reinterpret some of the foregoing results and comment on their implications for the higher group cohomology $H^j(G, U \otimes_k k[V])$.

Given a finitely generated graded integral domain R over the field k and a finite group Γ, we let Γ act trivially on R and denote by $R(\Gamma)$**-mod** the abelian category of finitely generated non-negatively graded $R(\Gamma)$-modules M. The assignment $M \mapsto \mathrm{Tor}^R(M, k)$ is a functor from the category $R(\Gamma)$-**mod** to the category **bigraded-**$k(\Gamma)$**-mod** of bigraded-$k(\Gamma)$-modules that are finite-dimensional in each bidegree. One can compose this with the forgetful functor **bigraded-**$k(\Gamma)$**-mod** $\to$ **graded-**$k(\Gamma)$**-mod** that forgets the internal grading by taking the direct sum over it, and leaves the homological grading. Theorem 2.5.2(iii) asserts that the composite of these two functors followed by taking the alternating sum over the homological grading yields a well-defined homomorphism of Grothendieck rings $\mathbf{R}(R(\Gamma)) \to \mathbf{R}(k(\Gamma))$, which sends $[M] \mapsto \psi^{-1}[K \otimes_R M]$, where K is the field of fractions of R and ψ is induced from the inclusion $k \hookrightarrow K$.

Now we place ourselves in the context of Theorem 1.1.1 (iii) where $R = k[V]^G$ and we are considering the *relative invariants* functor in the form $k(\Gamma \times G)$**-mod** $\to R(\Gamma)$**-mod** that sends $U \mapsto (U \otimes_k k[V])^G$. Then Theorem 1.1.1 (iii) says that if we follow this by the composite functor described above, we get a well-defined homomorphism $\mathbf{R}(k(\Gamma \times G)) \to \mathbf{R}(k(\Gamma))$ which sends $[U] \mapsto [U]$, that is, it coincides with the restriction homomorphism that simply forgets the $k(G)$-structure.

This is a bit surprising, as the relative invariants functor is not exact; it is the case $j = 0$ of a family of (generally nontrivial) group cohomology functors

$$U \mapsto M_j(U) := H^j(G, U \otimes_k k[V])$$

for $j \geq 0$, which measure the inexactness of taking relative invariants. In fact, it is not hard to show using the same ideas as above, that for any strictly positive j, when one follows $M_j(-)$ by the composite functor and takes an alternating sum (i.e., Euler characteristic) as discussed above, it induces the *zero* homomorphism $\mathbf{R}(k(\Gamma \times G)) \to \mathbf{R}(k(\Gamma))$. That is, one has the following conclusion.

Theorem 2.7.1. *Let G be a finite subgroup of $GL(V)$, let $R = k[V]^G$ and let U be any finite-dimensional $k(G \times \Gamma)$-module. For all $j \geq 0$, let*

$$M_j(U) := H^j(G, U \otimes_k k[V])$$

be the j^{th} cohomology group of G with coefficients in $U \otimes_k k[V]$, considered as a $R(\Gamma)$-module. Then for strictly positive j one has

$$\left. \sum_{i \geq 0} (-1)^i \ \mathrm{Tor}_i^R(M_j(U), k) \right|_{t=1} = 0$$

in $\mathbf{R}(k(\Gamma))$

Before proving this, we quote a standard fact about the behavior of group cohomology under change of ground rings, which will also be of use further on in the proof of Proposition 3.3.1.

Proposition 2.7.2. *Let G be a finite group and let $R \to S$ be a homomorphism of commutative rings which is flat, so that $M \to S \otimes_R M$ is an exact functor from R-modules to S-modules. Then for any $R(G)$-module M, one has*

$$S \otimes_R H^j(G, M) \cong H^j(G, S \otimes_R M)$$

as S-modules.

Proof. See [12, Sec. 10.3, Prop. 7]. □

Proof of Theorem 2.7.1. Applying Theorem 2.5.2(iii), and the remarks in Section 2.4 concerning extension of scalars, it suffices for us to show that

$$K \otimes_R H^j(G, U \otimes_k k[V]) = 0 \text{ for } j > 0,$$

where K is the field of fractions of R. Note that $K \otimes_R (-)$ is exact, because it is a localization, so Proposition 2.7.2 implies the first isomorphism in the following string of isomorphisms and equalities.

$$\begin{aligned} K \otimes_R H^j(G, U \otimes_k k[V]) &\cong H^j(G, K \otimes_R (U \otimes_k k[V])) \\ &\cong H^j(G, U \otimes_k k(V)) \\ &\cong H^j(G, U \otimes_k KG) \\ &\cong H^j(G, K \otimes_k (U \otimes_k k(G))) \\ &= 0 \end{aligned}$$

The equality on the second line is by definition, while the next three isomorphisms appeared as $\overset{(2)}{\cong}, \overset{(3)}{\cong}, \overset{(4)}{\cong}$ in Section 2.6. The last vanishing assertion comes from the fact that $U \otimes_k k(G)$ is always free as a $k(G)$-module, and hence $K \otimes_k (U \otimes_k k(G)))$ is free as a $K(G)$-module, so its higher cohomology will vanish. □

We close this section by noting that whereas we have been dealing with the fixed points of G and its derived functors, we could instead have worked with the fixed *quotients* by the action of G. Specifically, the same arguments that we have used in the proof of Theorem 1.1.1 can also be used if we replace M by $M' := U \otimes_{k(G)} k[V]$, namely the fixed quotient of G acting on $U \otimes_k k[V]$ rather the fixed points which we have used before. Of course, when $|G|$ is invertible in k these constructions are isomorphic, and so they are both valid interpretations of the notion of relative invariants in the modular case. With this definition of M' the crucial chain of equations in the proof of Theorem 1.1.1 becomes

$$\begin{aligned} K \otimes_R M' &= k(V)^G \otimes_{k[V]^G} (U \otimes_{k(G)} k[V]) \\ &= U \otimes_{k(G)} (k[V] \otimes_{k[V]^G} k(V)^G) \\ &\cong (U \otimes_{k(G)} k(V)) \\ &\cong (U \otimes_{k(G)} KG) \\ &\cong K \otimes_k (U \otimes_{k(G)} k(G)) \\ &\cong K \otimes_k U \end{aligned}$$

and this shows that it is valid to replace M by M' in the statement of Theorem 1.1.1.

3. Proof of Theorems 1.2.1 and 1.3.1

Let us recall the setting and statement of Theorem 1.3.1. We suppose k is an arbitrary field, G, Γ, and C finite groups, and V a finite dimensional $(k(G), k(C))$-bimodule on which G acts faithfully, so $G \subset GL(V)$. We regard $V, k[V]$ and $k[V]^G$ as trivial $k(\Gamma)$-modules.

Suppose there is a Noether normalization $R \subset k[V]^G$ that is stable under the action of C on $k[V]$. Further suppose that one has a vector v in V

such that the fiber $\Phi_v := \phi^{-1}(\phi(v))$ containing v for the map $\phi : V \to \operatorname{Spec}(R)$ both carries a free (but not necessarily transitive) G-action, and that this fiber Φ_v is stable under C. Denote by $\mathfrak{m}_{\phi(v)}$ the maximal ideal in R corresponding to $\phi(v)$ in $\operatorname{Spec}(R)$ and introduce the coordinate ring of Φ_v namely, $A(\Phi_v) = k[V]/\mathfrak{m}_{\phi(v)}k[V]$, which is a $k(C)$-module.

Let U be a finite-dimensional $(k(\Gamma), k(G))$-bimodule which we regard as a trivial $k(C)$-module.

In this situation Theorem 1.3.1 asserts that the relative invariants $M := (U \otimes_k k[V])^G$ satisfy the equation

$$\sum_{i\geq 0}(-1)^i \ \left[\operatorname{Tor}_i^R(M,k)\right] = \left[(U \otimes_k A(\Phi_v))^G\right].$$

in $\mathbf{R}(k(\Gamma \times C))$. Note that Hilbert's syzygy theorem tells us that the sum is finite since R is a polynomial algebra.

In the subsections that follow we make various reductions leading up to the proof of Theorem 1.3.1, with the goal of separating out the different ideas involved.

3.1. **Reduction 1: Removing the Γ-action.** The goal here is to prove the following lemma.

Lemma 3.1.1. *Theorem 1.3.1 follows from its special case where* Γ *acts trivially on* U*; that is, the case with only a* C*-action and no* Γ*-action.*

This will essentially be a consequence of the Brauer theory reviewed in Section 2.4, applied to the $k(\Gamma)$-modules $(U \otimes_k k[V])^G$ and $(U \otimes_k A(\Phi_v))^G$.

Given a p-regular element γ in Γ, let $\tilde{G} := \langle\gamma\rangle \times G$. For any $(k(\Gamma), k(G))$-bimodule W, which we regard as a left $k(\Gamma \times G)$-module, its G-fixed subspace W^G is a semisimple $k\langle\gamma\rangle$-module, and one can express its $k\langle\gamma\rangle$-isotypic direct sum decomposition in terms of $\tilde{G}$-fixed subspaces:

$$W^G \cong \bigoplus_{j\in\mathbb{Z}/d\mathbb{Z}} (W \otimes_k U_j)^{\tilde{G}} \tag{3.1.1}$$

If W happens also to have an action of C commuting with the actions of Γ and G this becomes a direct sum of $k(C)$-modules.

Proof of Lemma 3.1.1. Apply the preceding discussion to the $(k(G), k(\Gamma))$-bimodules $W = U \otimes_k A(\Phi_v)$ and $W = U \otimes_k k[V]$. Using the fact that functors like tensor product and Tor commute with direct sums, along with the Brauer theory from Section 2.4, one sees that Theorem 1.3.1 is equivalent to showing for each p-regular element $\gamma \in \Gamma$

$$\sum_{i\geq 0}(-1)^i \ \left[\operatorname{Tor}^R((\tilde{U}_j \otimes_k k[V])^{\tilde{G}}, k)\right] = \left[(\tilde{U}_j \otimes_k A(\Phi_v))^{\tilde{G}}\right] \tag{3.1.2}$$

in $\mathbf{R}(k(C))$, where U_j ranges over the simple $k(\langle\gamma\rangle)$-modules and $\tilde{U}_j := U \otimes_k U_j$. However, the equality (3.1.2) is an instance of Theorem 1.3.1 with trivial Γ-action: Simply replace G by $\tilde{G}$, and U by $\tilde{U}_j$. □

3.2. **Reduction 2: Replacing Special Fiber with General Fiber.** A key idea in the proof of Theorem 1.3.1 goes back to Borho and Kraft [5], before that to Kostant [14], and perhaps earlier: Given the finite ramified cover $\phi : V \to \mathrm{Spec}(R)$, one should compare the group actions on the special fiber Φ_0 and its coordinate ring $k[V]/R_+k[V]$ with the (often better understood) actions on a more general fiber Φ_v and its coordinate ring $k[V]/\mathfrak{m}_{\phi(v)}k[V]$.

To set this up, we return to the situation of Section 2 with an R-module M and compatible finite group $\mathcal{G}$ acting as above. Then $k = R/R_+$ carries the trivial R-module and $k(\mathcal{G})$-action. Let k' be a different, possibly non-graded, R-module structure on the field k. In other words, $k' = R/\mathfrak{m}'$ where $\mathfrak{m}'$ is some (generally inhomogeneous) maximal ideal of R which happens to be $\mathcal{G}$-stable. Note that the action of $\mathcal{G}$ on k' remains trivial, since k' is spanned by the image of 1, on which $\mathcal{G}$ acts trivially.

Proposition 3.2.1. *Let $\mathcal{G}$ be a finite group, R an $\mathbb{N}$-graded Noetherian algebra over the field k and M a finitely generated R-module. Assume that both R and M have compatible $\mathcal{G}$-actions and that $\mathrm{hd}_R(M)$ is finite. Denote by $\mathfrak{m}$ the (tautological) maximal ideal R_+ of R (so $R/\mathfrak{m} \cong k$ is the tautological R-module structure on k), and by $\mathfrak{m}'$ is some (possibly inhomogenous) maximal ideal of R which is $\mathcal{G}$-stable. Set $k' = R/\mathfrak{m}'$.*

Then one has the following (ungraded) equality in $\mathbf{R}(k(\mathcal{G}))$ of two (finite) sums:

$$\sum_{i,j\geq 0} (-1)^i [\mathrm{Tor}_i^R(M,k)_j] = \sum_{i,j\geq 0} (-1)^i [\mathrm{Tor}_i^R(M,k')_j].$$

Proof. Compute either of the two Tor's by starting with a finite (not necessarily minimal) free R-resolution $\mathcal{F}$ produced as in Proposition 2.3.1(iii), tensoring over R with k or k', and then taking the homology of either $\mathcal{F} \otimes_R k$ or $\mathcal{F} \otimes_R k'$. Each term $F_i \otimes_R k$ or $F_i \otimes_R k'$ is a finite-dimensional k-vector space, and taking alternating sums in $\mathbf{R}(k(\mathcal{G}))$ gives the following equalities

$$\sum_{i,j\geq 0} (-1)^i \left[\mathrm{Tor}_i^R(M,k)_j\right] = \sum_{i\geq 0} (-1)^i [F_i \otimes_R k]$$
$$\sum_{i,j\geq 0} (-1)^i \left[\mathrm{Tor}_i^R(M,k')_j\right] = \sum_{i\geq 0} (-1)^i [F_i \otimes_R k'].$$

Note the sums are finite because M is finitely generated and $\mathrm{hd}_R(M)$ is finite.

It therefore suffices to show that as $k(\mathcal{G})$-modules only (disregarding their R-module structure), one has $[F_i \otimes_R k] = [F_i \otimes_R k']$ in $\mathbf{R}(k(\mathcal{G}))$, which we prove by a filtration argument. Given a homogeneous R-basis $\{e_\alpha\}$ for F_i, one has filtrations $\mathcal{A}, \mathcal{A}'$ on the two k-vector spaces $F_i \otimes_R k, F_i \otimes_R k'$ defined as follows: let $\mathcal{A}_j, \mathcal{A}'_j$ be the k-span of those k-basis elements $e_\alpha \otimes_R 1$ in which $\deg(e_\alpha) \leq j$. Since $\mathcal{G}$ acts in a grade-preserving fashion, it will preserves these filtrations. We claim that there is also a $k(\mathcal{G})$-module isomorphism $\mathcal{A}_j/\mathcal{A}_{j-1} \to \mathcal{A}'_j/\mathcal{A}'_{j-1}$ sending the k-basis element $e_\alpha \otimes 1$ to the k-basis element $e_\alpha \otimes 1$. To check that this isomorphism is $\mathcal{G}$-equivariant, given

$g \in \mathcal{G}$, let $g(e_\alpha) = \sum_\beta r_{\beta,\alpha}(g) e_\beta$ for some homogeneous elements $r_{\beta,\alpha}(g)$ in R. One then has the same computation in either of $\mathcal{A}_j/\mathcal{A}_{j-1}$ or $\mathcal{A}'_j/\mathcal{A}'_{j-1}$:

$$g(e_\alpha \otimes 1) = \sum_{\beta : \deg(e_\beta) = \deg(e_\alpha)} r_{\beta,\alpha}(g)\,(e_\beta \otimes 1)$$

Note that in this last sum, the coefficient $r_{\beta,\alpha}(g)$ in R represents the same element in the quotient fields k or k', since it must be of degree zero by homogeneity considerations. □

We comment that in the last part of the proof of Proposition 3.2.1 we do not necessarily get an isomorphism of $k(\mathcal{G})$-modules, as can be seen by considering an example where $R = \mathbb{F}_2[x]$ acted on trivially by a cyclic group $\mathcal{G}$ of order 2. We may take $F_i = Re_0 \oplus Re_1$ to be free of rank 2, where e_0 lies in degree 0 and e_1 lies in degree 1. Let $\mathcal{G}$ act on F_i via the matrix

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

and let $k' = R/(x-1)$. Then $\mathcal{G}$ acts on $\mathcal{F} \otimes_R k$ via

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and on $\mathcal{F} \otimes_R k'$ via

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

These two actions are non-isomorphic.

The particular case of Proposition 3.2.1 that will interest us most is where one has a finite group $G \subseteq GL(V)$, with $R \subseteq k[V]^G$ an integral extension of graded algebras. Then given any finite-dimensional $k(G)$-module U, as in the Introduction, one can form the R-module $U \otimes_k k[V]$, with diagonal $k(G)$-action, having G-fixed subspace $M := (U \otimes_k k[V])^G$ which retains the structure of an R-module (because G acts trivially on R).

Proposition 3.2.2. *Let G be a finite group that acts on the finite dimensional k-vector spaces V and $R \subset k[V]^G$ a Noether normalization. Then for any finite dimensional $k(G)$-module U the module of relative invariants $M := (U \otimes_k k[V])^G$ is finitely-generated as an R-module.*

Proof. Recall the tower of integral extensions

$$R \subseteq k[V]^G \subseteq k[V].$$

Note that $U \otimes_k k[V]$ is finitely-generated as a $k[V]$-module, hence also finitely-generated as a $k[V]^G$-module. Hence it is a Noetherian $k[V]^G$-module, and its $k[V]^G$-submodule $M = (U \otimes_k k[V])^G$ will be Noetherian, that is, finitely-generated over $k[V]^G$. But then M is also finitely-generated over R. □

In the situation of Theorem 1.3.1, with R, M, Γ, C as defined there, choose

$$k' = k_v := R/\mathfrak{m}_{\phi(v)}.$$

Note $\mathrm{hd}_R(M)$ is finite and bounded above by $\dim_k(V)$ via Hilbert's syzygy theorem since R is a polynomial algebra. Thus Proposition 3.2.1 together with Lemma 3.1.1 show that to prove Theorem 1.3.1 reduces to showing

$$\sum_{m\geq 0}(-1)^i \ \left[\mathrm{Tor}_i^R(M, k_v)\right] = \left[(U \otimes_k A(\Phi_v))^G\right] \tag{3.2.1}$$

in $\mathbf{R}(k(C))$.

The plan for proving (3.2.1) is pretty clear: Try to prove the vanishing result

$$\mathrm{Tor}_i^R(M, k_v) = 0 \text{ for } i > 0,$$

and then try to identify

$$\mathrm{Tor}_0^R(M, k_v) \cong (U \otimes_k A(\Phi_v))^G$$

as $k(C)$-modules.

3.3. **Reduction 3: Working Locally via Completions.** Both parts in the aforementioned plan for proving (3.2.1) in Section 3.2 have the advantage that one can work with local and semi-local rings, by taking completions with respect to $\mathfrak{m}_{\phi(v)}$-adic topologies. Completion will be shown to commute with Tor, reducing the vanishing statement $\mathrm{Tor}_i^R(M, k_v) = 0$ for $i > 0$ to showing that the completion $\hat{M}$ of M is free as a module over the completion $\hat{R}$ of R. Working locally also allows us to take advantage of the Chinese Remainder Theorem for semi-local rings.

We begin with a review of some properties of completion in a general setting–much of this material can be found in [1, Chap. 10], [9, Chap. 7], [16, §8]. Let R be a commutative Noetherian ring, M an R-module, and $\mathfrak{l}$ any ideal of R. Denote by $\hat{R}$ the **completion** of R with respect to the $\mathfrak{l}$-adic topology, and $\hat{M} \cong \hat{R} \otimes_R M$ the corresponding completion of M.

Proposition 3.3.1. *Let R be a commutative Noetherian ring, M an R-module, $\mathfrak{l}$ an ideal of R, and $\mathcal{G}$ a finite group acting on M by R-module homomorphisms. Then taking fixed points commutes with $\mathfrak{l}$-adic completion: $\left(\hat{M}\right)^{\mathcal{G}} \cong \widehat{M^{\mathcal{G}}}$ that is,*

$$(\hat{R} \otimes_R M)^{\mathcal{G}} \cong \hat{R} \otimes_R M^{\mathcal{G}}$$

as $\hat{R}$-modules.

Proof. Completion is an exact functor from R-modules to $\hat{R}$-modules, so commutes with the cohomology functors $H^j(\mathcal{G}, -)$ by Proposition 2.7.2. In particular, this holds for $j = 0$, the fixed-point functor $(-)^{\mathcal{G}}$. ∎

If $\mathfrak{m}$ is a *maximal* ideal of R, then the ideal $\hat{\mathfrak{m}} := \mathfrak{m}\hat{R}$ is also maximal in $\hat{R}$, and the two residue fields are the same, viz.,

$$k_{\mathfrak{m}} = R/\mathfrak{m} = \hat{R}/\hat{\mathfrak{m}}.$$

Thus $k_{\mathfrak{m}}$-vector spaces can be regarded as both R-modules and $\hat{R}$-modules. Furthermore, $\hat{R}$ is a *local* ring. The next proposition exploits this to translate the vanishing of Tor_i for all $i > 0$ into freeness of the completed module.

Proposition 3.3.2. *Let $\mathfrak{m}$ be a maximal ideal of a commutative Noetherian ring R, and M an R-module. Then as $k_{\mathfrak{m}}$-vector spaces one has*

$$\mathrm{Tor}_i^{\hat{R}}(\hat{M}, k_{\mathfrak{m}}) \cong \mathrm{Tor}_i^R(M, k_{\mathfrak{m}}).$$

Consequently $\mathrm{Tor}_i^R(M, k_{\mathfrak{m}})$ vanishes for all $i > 0$ if and only if $\hat{M}$ is a free $\hat{R}$-module.

Proof. The asserted isomorphism is a consequence of the string of isomorphisms that follows.

$$\begin{aligned}\mathrm{Tor}_i^{\hat{R}}(\hat{M}, k_{\mathfrak{m}}) &\cong \mathrm{Tor}_i^{\hat{R}}(\hat{R} \otimes_R \hat{M}, \hat{R} \otimes_R k_{\mathfrak{m}}) \\ &\cong \hat{R} \otimes_R \mathrm{Tor}_i^R(M, k_{\mathfrak{m}}) \\ &\cong \mathrm{Tor}_i^R(M, k_{\mathfrak{m}})\end{aligned}$$

The second of these uses the fact that $\hat{R} \otimes_R (-)$ is exact, and the last that $\mathrm{Tor}^R(M, N)$ is an $R/\mathrm{Ann}_R(N)$-module, so a vector space over $k_{\mathfrak{m}} = R/\mathfrak{m} = \hat{R}/\hat{\mathfrak{m}}$, and $\hat{R} \otimes_R k_{\mathfrak{m}} = k_{\mathfrak{m}}$.

Since $\hat{R}$ is a local ring with residue field $k_{\mathfrak{m}}$, the $\hat{R}$-module $\hat{M}$ is free if and only if $\mathrm{Tor}_i^{\hat{R}}(\hat{M}, k_{\mathfrak{m}}) = 0$ for $i > 0$. So the previous isomorphism implies $\mathrm{Tor}_i^R(M, k_{\mathfrak{m}}) = 0$ for all $i > 0$. □

3.4. **(Semi-)local Analysis of the Fibers.** Let k be a field and V a finite-dimensional k-vector space. Suppose that G is a finite subgroup of $GL(V)$ and $R \subset k[V]^G$ a Noether normalization, so we have integral extensions of graded algebras

$$R \subseteq k[V]^G \subseteq k[V].$$

Let U be a finite-dimensional $(k(\Gamma), k(G))$-bimodule for some finite group Γ with module of relative invariants $M := (U \otimes_k k[V])^G$. Suppose that there exists a vector $v \in V$ whose fiber Φ_v for the composite ϕ in the tower of ramified coverings

$$V \to V/G \to \mathrm{Spec}(R)$$

is permuted freely (but not necessarily transitively) by G. In this section we let Γ act trivially on V and ignore all C-actions. The C-actions will be put back in Section 3.5.

For any $w \in V$, denote by k_w the $k[V]$-module structure on k defined by $f \cdot \alpha := f(w)\alpha$ for all $f \in k[V]$ and $\alpha \in k$. The corresponding maximal ideal of polynomials vanishing at w is denoted by $\mathfrak{m}_w \subset k[V]$. By restriction we also consider k_w as module over $k[V]^G, R$, etc. The notations for the corresponding maximal ideals of $k[V]^G$ and R are

$$\mathfrak{m}_{Gw} = \mathfrak{m}_w \cap k[V]^G$$
$$\mathfrak{m}_{\phi(w)} = \mathfrak{m}_w \cap R.$$

The ideal $\mathfrak{m}_{Gw}$ can also be characterized as the G-invariant polynomials that vanish at the G-orbit Gw regarded as a point of V/G. Likewise, $\mathfrak{m}_{\phi(w)}$ can be characterized as the polynomials in R that vanish on the whole fiber

$$\Phi_w := \phi^{-1}(\phi(w)) = \{u \in V : \phi(u) = \phi(w)\},$$

and the generators are easy to describe: If $R = k[f_1, \ldots, f_m]$ then

$$\mathfrak{m}_{\phi(w)} = (f_1 - f_1(w), f_2 - f_2(w), \ldots, f_m - f_m(w))R.$$

Let $\hat{R}$ denote the complete local ring obtained by completing R at the maximal ideal $\mathfrak{m}_{\phi(w)}$.

An important property of completion, the Chinese remainder theorem (see e.g., [9, Corollary 7.6], [16, Theorem 8.15]), gives cartesian product decompositions of complete semi-local rings. Thus in the context just described one has the following table in which the first column lists various R-modules M, the second column lists their associated semi-local completed $\hat{R}$-modules $\hat{M} := \hat{R} \otimes_R M$ along with Chinese remainder theorem isomorphisms, and the third column lists their quotient k_w-modules , viz., $M/\mathfrak{m}_{\phi(w)}M \cong \hat{M}/\hat{\mathfrak{m}}_{\phi(w)}\hat{M}$.

R-module M	$\hat{R}$-module $\hat{M}$	k_w-module $M/\mathfrak{m}_{\phi(w)}M$
$A := k[V]$	$\hat{A} \cong \prod_{w \in \Phi_w} \hat{A}_w$	$A(\Phi_w)$
$B := k[V]^G$	$\hat{B}(= \hat{A}^G) \cong \prod_{Gw \in \Phi_w/G} \hat{B}_{Gw}$	$B(\Phi_w/G)$
R	$\hat{R}$	k_w

Table 3.4.1: Chinese Remainder Table

The heart of the matter now lies in using this to prove the following lemma.

Lemma 3.4.1. *Let k be a field and V a finite-dimensional k-vector space. Suppose that G is a finite subgroup of $GL(V)$ and $R \subset k[V]^G$ a Noether normalization. Assume that there is a $v \in V$ whose fibre Φ_v carries a free G-action. Let U be a finite-dimensional $(k(\Gamma), k(G))$-bimodule for some finite group Γ with module of relative invariants $M := (U \otimes_k k[V])^G$ considered as a $B(\Gamma)$-module in the notations of Table 3.4.1. Then with the notations of that table the following hold.*

(i) *As $\hat{B}(\Gamma)$-modules, one has isomorphisms*

$$\hat{M} \cong (U \otimes_k \hat{B}(G))^G \cong \hat{B} \otimes_k (U \otimes_k k(G))^G.$$

In particular, $\hat{A} \cong \hat{B}(G)$ *as* $\hat{B}(G)$*-modules; this is the special case* $U = k(G)$.

(ii) $\hat{M}$ *is a free* $\hat{R}$*-module, and hence*

$$\left(\mathrm{Tor}_i^R(M,k) =\right) \mathrm{Tor}_i^{\hat{R}}(\hat{M},k) = 0 \text{ for } i > 0.$$

(iii) *As* $k_v(\Gamma)$*-modules there are isomorphisms*

$$\mathrm{Tor}_0^R(M,k) \cong (U \otimes_k A(\Phi_v))^G \cong (U \otimes_k B(\Phi_v/G)(G))^G.$$

In particular, $A(\Phi_v) \cong B(\Phi_v/G)(G)$ *as* $B(\Phi_v/G)(G)$*-modules; this is the special case* $U = k(G)$.

Proof. We begin with some preparations making use of the isomorphism $\hat{A} \cong \prod_{w\in\Phi_v} \hat{A}_w$ from Table 3.4.1. Here the right side has componentwise multiplication, and has a $k(G)$-module structure given via the isomorphisms $\hat{A}_w \xrightarrow{g} \hat{A}_{gw}$ which permute the factors. As a consequence of the assumption that G acts freely on Φ_v, there is a decomposition of the fiber

$$\Phi_v = Gw_1 \sqcup \cdots \sqcup Gw_r$$

into free G-orbits. This gives two ways to regroup the factors, viz.,

$$\hat{A} \cong \prod_{g\in G} \underbrace{\left(\prod_{i=1}^r \hat{A}_{gw_i}\right)}_{:=\hat{A}_g} \qquad \text{and} \qquad \hat{A} \cong \prod_{i=1}^r \underbrace{\left(\prod_{g\in G} \hat{A}_{gw_i}\right)}_{:=\hat{A}_{Gw_i}}.$$

Note that here $\hat{A}_g = g(\hat{A}_e)$ where $\hat{A}_e = \prod_{i=1}^r \hat{A}_{w_i}$.

View $\hat{A}$ as an $\hat{A}_e$-algebra and $\hat{A}_{Gw_i}$ as a $\hat{A}_{w_i}$-algebra via the diagonal embeddings

$$\begin{array}{lcll} \hat{A}_e & \xrightarrow{\Delta} & \prod_{g\in G} \hat{A}_g & \cong \hat{A} \\ \hat{A}_{w_i} & \xrightarrow{\Delta} & \prod_{g\in G} \hat{A}_{gw_i} & \cong \hat{A}_{Gw_i} \end{array}$$

defined by mapping a to the product $(g(a))_{g\in G}$. These embeddings give ring isomorphisms of $\hat{A}_e$ and $\hat{A}_{Gw_i}$ onto their diagonal images, which are exactly the G-invariant subrings of the relevant completions, or the completions of the relevant G-invariant subrings by Proposition 3.3.1. To wit,

$$\begin{array}{llll} \hat{A}_e & \cong \Delta(\hat{A}_e) & \cong \hat{A}^G & \cong \hat{B} \\ \hat{A}_{w_i} & \cong \Delta(\hat{A}_{w_i}) & \cong (\hat{A}_{Gw_i})^G & \cong \hat{B}_{Gw_i} \end{array} \tag{3.4.1}$$

Next consider the group algebras $\hat{A}_e(G)$ and $\hat{A}_{w_i}(G)$ for the group G, with either $\hat{A}_e$ or $\hat{A}_{w_i}$ as coefficients; in either case, denote by t_g the basis element corresponding to the element g in G. We claim that there are isomorphisms of $\hat{A}_e(G)$-modules and $\hat{A}_{w_i}(G)$-modules

$$\begin{array}{lcll} \hat{A}_e(G) & \xrightarrow{\alpha} & \prod_{g\in G} \hat{A}_g & \cong \hat{A} \\ \hat{A}_{w_i}(G) & \xrightarrow{\alpha} & \prod_{g\in G} \hat{A}_{gw_i} & \cong \hat{A}_{Gw_i} \end{array} \tag{3.4.2}$$

defined in both cases by

$$at_g \longmapsto g(a)e_g$$

where e_g is the standard basis vector/idempotent corresponding to the factor in the product indexed by g, and where a lies either in $\hat{A}_e$ or $\hat{A}_{w_i}$ (so that $g(a)$ lies either in $g(\hat{A}_e) = \hat{A}_g$ or $g(\hat{A}_{w_i}) = \hat{A}_{gw_i}$). The inverse isomorphism α^{-1} in either case is defined by

$$(a^{(g)})_{g\in G} = \sum_{g\in G} a^{(g)} e_g \quad \longmapsto \quad \sum_{g\in G} g^{-1}a^{(g)}t_g.$$

With this preparation, we can start to prove the assertions of the Lemma, beginning with assertion (i). In light of the first isomorphism $\hat{A}_e \cong \hat{B}$ in (3.4.1) the first isomorphism in (3.4.2) shows $\hat{A} \cong \hat{B}(G)$ as a $\hat{B}(G)$-modules. Since G acts trivially on R, using Proposition 3.3.1 again, one has

$$\begin{aligned} \hat{M} := \hat{R} \otimes_R ((U \otimes_k A)^G) &\cong (\hat{R} \otimes_R (U \otimes_k A))^G \\ &\cong (U \otimes_k \hat{A})^G \\ &\cong (U \otimes_k \hat{B}(G))^G \\ &\cong \hat{B} \otimes_k (U \otimes_k k(G))^G \end{aligned}$$

as $\hat{B}$-modules. However, these are also $\hat{B}(\Gamma)$-module isomorphisms since the Γ-action occurs entirely in the U factor and acts trivially on R, B, A and their completions.

For (ii), it suffices to show that $\hat{M}$ is a free $\hat{R}$-module, and then to apply Proposition 3.3.2(ii). Since (i) implies $\hat{M}$ is a free $\hat{B}$-module it suffices to show that $\hat{B}$ is a free $\hat{R}$-module. From Table 3.4.1, one has $\hat{B} \cong \prod_{i=1}^r \hat{B}_{Gw_i}$, and hence it is enough to verify that each $\hat{B}_{Gw_i}$ is a free $\hat{R}$-module. Note that the ring $\hat{B}_{Gw_i}$ is a finite extension of the ring $\hat{R}$. It turns out that both of these are regular local rings, because they are isomorphic to completions of polynomial algebras at maximal ideals ([9, Corollary 19.14]): in the case of $\hat{R}$ this is due to the assumption that R is polynomial, and in the case of $\hat{B}_{Gw_i}$ this is due to the second isomorphism $\hat{B}_{Gw_i} \cong \hat{A}_{w_i}$ of (3.4.1). Hence according to the Auslander-Buchsbaum Theorem [9, Theorem 19.9], using dim(—) to indicate Krull dimension of — , we obtain

$$\begin{aligned} \mathrm{hd}_{\hat{R}}(\hat{B}_{Gw_i}) &= \dim(\hat{R}) - \mathrm{depth}_{\hat{R}}(\hat{B}_{Gw_i}) \\ &= \dim(\hat{R}) - \dim(\hat{B}_{Gw_i}) \\ &= \dim(\hat{R}) - \dim(\hat{R}) \\ &= 0. \end{aligned}$$

The second equality here is due to the fact that regular local rings are Cohen-Macaulay, so their depth and Krull dimension are the same, while the third follows from the fact that $\hat{B}_{Gw_i}$ is a finite extension of $\hat{R}$. Thus $\hat{B}_{Gw_i}$ is $\hat{R}$-free, and hence so is $\hat{M}$.

For (iii), note that

$$\begin{aligned}
\mathrm{Tor}_0^R(M,k) &\cong M/\mathfrak{m}_{\phi(v)}M \\
&\cong \hat{M}/\hat{\mathfrak{m}}_{\phi(v)}\hat{M} \\
&\cong \left(U\otimes_k \hat{A})^G\right)/\hat{\mathfrak{m}}_{\phi(v)}\left(U\otimes_k \hat{A})^G\right) \\
&\cong \left(U\otimes_k \left(\hat{A}/\hat{\mathfrak{m}}_{\phi(v)}\hat{A}\right)\right)^G \\
&\cong \left(U\otimes_k \left(A/\hat{\mathfrak{m}}_{\phi(v)}A\right)\right)^G =: (U\otimes_k A(\Phi_v))^G.
\end{aligned}$$

This gives the first isomorphism in (iii). For the second, note that

$$\begin{aligned}
\hat{A}/\hat{\mathfrak{m}}_{\phi(v)}\hat{A} &\cong \hat{B}(G)/\hat{\mathfrak{m}}_{\phi(v)}\hat{B}(G) \\
&\cong (\hat{B}/\hat{\mathfrak{m}}_{\phi(v)}\hat{B})(G) \\
&\cong (B/\hat{\mathfrak{m}}_{\phi(v)}B)(G) \\
&= B(\Phi_v/G)(G)
\end{aligned}$$

□

3.5. **Finishing the proofs: Incorporating the C-Action.** Assertions (ii) and (iii) of Lemma 3.4.1 complete the proof of Theorem 1.3.1, except that we have not yet accounted for the C-action. We rectify this here, and explain how Theorem 1.3.1 implies Theorem 1.2.1.

So we assume there is also a finite subgroup $C \subset GL(V)$, commuting with G, that preserves R and the maximal ideal $\mathfrak{m}_{\phi(v)}$, making C act on the fiber Φ_v. One then has compatible C-actions on

$$\begin{array}{lllll}
R & \subseteq & B = k[V]^G & \subseteq & A = k[V] \\
\hat{R} & \subseteq & \hat{B} & \subseteq & \hat{A} \\
k_v & \subseteq & B(\Phi_v/G) & \subseteq & A(\Phi_v).
\end{array}$$

We wish to describe these actions more explicitly under the assumption that the G-orbits $\{Gw_i\}_{i=1}^r$ in Φ_v/G are all regular. Note that C permutes these G-orbits, since it commutes with G, and if $c \in C$ stabilizes some G-orbit Gw_i, then there will be a unique element $g_{c,w_i} \in G$ for which

$$cw_i = g_{c,w_i}w_i. \tag{3.5.1}$$

One checks that this element g_{c,w_i} depends only on the choice of the representative w_i for the orbit Gw_i up to conjugacy as follows. First

$$g_{c,hw_i} = hg_{c,w_i}h^{-1}.$$

However, once a choice of representative w_i is made, one has

$$cgw_i = gcw_i = gg_{c,w_i}w_i$$

for all $g \in G$.

Also recall that for each $i = 1, 2, \ldots, r$, there is an isomorphism (see (3.4.1) of §3.4) of the completed local rings $\hat{A}_{w_i} \cong \hat{B}_{Gw_i}$, which are finite extensions of the local ring $\hat{R}_{\mathfrak{m}_{\phi(v)}}$. Let

$$\begin{aligned} B_{(Gw_i)} &:= B_{Gw_i}/\mathfrak{m}_{\phi(v)} B_{Gw_i} \\ &\cong \hat{B}_{Gw_i}/\hat{\mathfrak{m}}_{\phi(v)} \hat{B}_{Gw_i} \\ &\cong \hat{A}_{w_i}/\hat{\mathfrak{m}}_{\phi(v)} \hat{A}_{w_i} \\ &\cong A_{w_i}/\mathfrak{m}_{\phi(v)} A(w_i). \end{aligned}$$

This ring is a finite-dimensional k_v-vector space; it may be viewed either as the coordinate ring for the (possibly non-reduced) structure on the fiber Φ_v local to the point w_i, as a subscheme of V, or for the structure on the orbit space Φ_v/G local to the orbit Gw_i, as a subscheme of V/G.

Lemma 3.5.1. *Assume the notation and hypotheses of Lemma 3.4.1. Further assume, as in this section, that there is a finite subgroup $C \subset GL(V)$ which preserves the Noether normalization R and the fiber Φ_v, Choose for each G-orbit Gw_i within Φ_v a representative w_i, and define g_{c,w_i} in G via (3.5.1) whenever $cGw_i = Gw_i$.*

Then the isomorphism of $B(\Phi_v/G)(G)$-module asserted in part (iii) of that lemma, viz.,

$$\mathrm{Tor}_0^R(M,k) \cong (U \otimes_k A(\Phi_v))^G \cong (U \otimes_k B(\Phi_v/G)(G))^G.$$

is also a $k(C)$-module isomorphism, obtained by using the $k(C)$-module structure induced from the following isomorphisms:

$$A(\Phi_v) \cong B(\Phi_v/G)(G) \cong \prod_{i=1}^{r} B_{Gw_i}(G).$$

If $c \in C$ has $cGw_i = Gw_j$ for $j \neq i$, then there is a ring isomorphism $B_{Gw_i} \xrightarrow{c} B_{Gw_j}$ that extends to a ring isomorphism $B_{Gw_i}(G) \xrightarrow{c} B_{Gw_j}(G)$.

For $c \in C$ with $cGw_i = Gw_i$, the ring automorphism $B_{Gw_i} \xrightarrow{c} B_{Gw_i}$ extends to a ring automorphism

$$\begin{aligned} B_{Gw_i}(G) &\xrightarrow{c} B_{Gw_i}(G) \\ at_g &\longmapsto c(a)t_{gg_{c,w_i}}. \end{aligned}$$

Consequently, there is the following identity relating Brauer character values:

$$\chi_{\mathrm{Tor}_0^R(M,k)}(c) = \sum_{i:cGw_i=Gw_i} \chi_{B_{Gw_i}}(c) \cdot \chi_U(g_{c,w_i}^{-1}). \tag{3.5.2}$$

Proof. The assertions will be derived by passing to the quotient by $\mathfrak{m}_{\phi(v)}$ from the analogous statement for the $k(C)$-module structures on the completed rings $\hat{A}, \hat{B}$, etc.

Note that the Chinese Remainder Theorem isomorphism

$$\hat{A} \cong \prod_{w \in \Phi_v} \hat{A}_w$$

translates the C-action on $\hat{A}$ to a C-action by isomorphisms $\hat{A}_w \xrightarrow{c} \hat{A}_{c(w)}$ permuting the factors on the right. From this, and the isomorphisms

$$\hat{B} \cong \prod_{i=1}^{r} \hat{B}_{Gw_i}$$

$$\hat{A} \cong \hat{B}(G) \cong \prod_{i=1}^{r} \hat{B}_{Gw_i}(G),$$

it is straightforward to check the assertions for the case when $c \in C$ has $cGw_i = Gw_j$ for $j \neq i$.

If $c \in C$ has $cGw_i = Gw_i$, then there is an automorphism c of $\hat{A}_{Gw_i} = \prod_{g \in G} \hat{A}_{gw_i}$, which acts by isomorphisms $\hat{A}_{gw_i} \xrightarrow{c} \hat{A}_{gg_{c,w_i}w_i}$ between the components. This action of c translates over to $\hat{A}_{w_i}(G)$ using the isomorphism α from the proof Lemma 3.4.1 which sends $at_g \overset{\alpha}{\mapsto} g(a)e_g$. So c sends $g(a)e_g$ to $cg(a)e_{gg_{c,w_i}}$, and this maps under α^{-1} to

$$(gg_{c,w_i})^{-1} cg(a) t_{gg_{c,w_i}} = g_{c,w_i}^{-1} g^{-1} cg(a) t_{gg_{c,w_i}} = g_{c,w_i}^{-1} c(a) t_{gg_{c,w_i}},$$

where the last equality uses the fact that C and G commute within $GL(V)$. Thus $c(at_g) = g_{c,w_i}^{-1} c(a) t_{gg_{c,w_i}}$.

From here one can translate the C-action to $\hat{B}_{Gw_i} \cong \prod_{g \in G} \hat{A}_{gw_i}$ using the diagonal embedding $\hat{A}_{w_i} \xrightarrow{\Delta} \hat{B}_{Gw_i}$ that maps $a \mapsto \Delta(a) = (g(a))_{g \in G}$. One then finds that $g_{c,w_i}^{-1} c(a) \overset{\Delta}{\mapsto} c(\Delta(a))$. In other words, a typical element bt_g in $\hat{B}_{Gw_i}(G)$ is sent by c to $c(b) t_{gg_{c,w_i}}$, as claimed.

The assertion about Brauer characters is then a consequence of Lemma 3.5.2 which follows. □

Lemma 3.5.2. *Given an element g_0 in the finite group G and a finite-dimensional $k(G)$-module U, let a cyclic group $C = \langle c \rangle$ act on U via $c(u) := g_0^{-1}(u)$, and let C act on $(k(G) \otimes_k U)^G$ via*

$$c\left(\sum_{g \in G} t_g \otimes u_g\right) := \sum_{g \in G} t_{gg_0} \otimes u.$$

Then $(k(G) \otimes_k U)^G \cong U$ as $k(C)$-modules, and consequently, if g_0 is p-regular, the Brauer character value for c acting on $(k(G) \otimes_k U)^G$ is

$$\chi_{(k(G) \otimes U)^G}(c) = \chi_U(g_0^{-1}).$$

Proof. The map

$$\begin{aligned} U &\longrightarrow (k(G) \otimes_k U)^G \\ u &\mapsto \sum_{g \in G} t_g \otimes g(u) \end{aligned}$$

is easily checked to give the desired $k(C)$-module isomorphism. □

Proof of Theorem 1.2.1. Assume that $k[V]^G$ is polynomial, and c in G is a regular element, so that $c(v) = \omega v$ for some vector v whose G-orbit Gv is free.

We wish to apply Theorem 1.3.1 with $R = k[V]^G$, so that Φ_v consists of only the regular G-orbit Gv (that is, $r = 1$ and $w_1 = v$ is the representative of the unique G-orbit on Φ_v). In this case, the local rings $\hat{R}_{\phi(v)} = \hat{B}_{Gv}$ are the same, and their quotient $B(Gv)$ by the maximal ideal $\mathfrak{m}_{\phi(v)} = \mathfrak{m}_{Gv}$ is the field $k_v \cong k$. Thus as $k(G)$-modules, one has $A(\Phi_v) \cong k(G)$, and Theorem 1.3.1 implies that

$$\sum_{i \geq 0} (-1)^i \left[\mathrm{Tor}^R(M, k)\right] = [(k(G) \otimes_k U)^G] = [U]$$

as $k(\Gamma)$-modules, where $M = (U \otimes_k k[V])^G$, for any finite-dimensional $k(G)$-module U and finite subgroup Γ of $\mathrm{Aut}_{k(G)} U$.

We wish to also take into account the action of a cyclic group $C = \langle \tau \rangle \subset \mathrm{Aut}_{k(G)} V$ whose generator τ acts as the scalar ω^{-1} on V. Then τ scales V^* by ω, and acts on the graded rings and modules $R = k[V]^G, k[V], M := (U \otimes k[V])^G$ by the scalar ω^j in their j^{th} homogeneous component, exactly as in the C-action considered in Theorem 1.2.1. Note that τ acts on $\Phi_v = Gv$ by $\tau(v) = c^{-1}(v)$ and more generally $\tau(g(v)) = \omega \cdot g(v) = gc^{-1}(v)$. Thus Lemma 3.5.1 shows that the $k(C)$-structure on $A(\Phi_v) \cong k(G)$ has $\tau(t_g) = t_{gc^{-1}}$, in agreement with the $k(G \times C)$-structure on $k(G)$ that appeared in Springer's Theorem, and Lemma 3.5.2 then shows that the $k(\Gamma \times C)$ structure on U agrees with the one that appears in Theorem 1.2.1. □

3.6. **Induced Modules and the Proof of Corollary 1.2.2.** We would like to apply Theorems 1.2.1 and 1.3.1 to modules M which are (relative) invariants for a subgroup H of G, rather than for G itself; in particular, we would like to draw conclusions about the H-invariant subring $k[V]^H$, as in Corollary 1.2.2.

This is made possible by a suitable notion of induction of $k(H)$-modules to $k(G)$-modules. Regard $k(G)$ as a $k(G \times H)$-module via the action

$$(g, h) \cdot t_{g'} := gg'h^{-1}.$$

Then given any finite-dimensional $k(H)$-module W, define its *induced* $k(G)$-module to be

$$\begin{aligned} \mathrm{Ind}_H^G(W) &:= \mathrm{Hom}_{k(H)}(k(G), W) \\ &= \{f \in \mathrm{Hom}_k(k(G), W) : f(t_{gh^{-1}}) = h(f(t_g))\}. \end{aligned}$$

That this is a $k(G)$-module follows from the equality $(g \cdot f)(t_{g'}) = f(t_{g^{-1}g'})$. We next explain how this construction converts relative invariants for G into relative invariants for H.

Proposition 3.6.1 (Frobenius Reciprocity)**.** *For finite-dimensional $k(G)$-modules V and $k(H)$-modules W,*

$$(V \otimes_k \operatorname{Ind}_H^G(W))^G \cong (\operatorname{Res}_H^G(V) \otimes_k W)^H.$$

Proof. This is a standard isomorphism which follows from [23, §3.3, Lemma 1] or [3, 3.3.1, 3.3.2], giving

$$(V \otimes_k \operatorname{Ind}_H^G(W))^G \cong (\operatorname{Ind}_H^G(\operatorname{Res}_H^G(V) \otimes_k W))^G \cong (\operatorname{Res}_H^G(V) \otimes_k W)^H.$$

as required. □

We will be particularly interested in the situation where the $k(H)$-module W is the trivial module, so that $\operatorname{Ind}_H^G(W) = \operatorname{Ind}_H^G(k)$ is the permutation module for G on the cosets $H \backslash G$, and the H-relative invariants are simply the H-invariants, namely, $(V \otimes_k W)^H = (V \otimes_k k)^H = V^H$. In this situation, the isomorphism in Proposition 3.6.1 becomes a $k(\Gamma)$-module isomorphism for the action of the group $\Gamma := N_G(H)/H$. This is a special case of the following result.

Proposition 3.6.2. *Consider nested subgroups $H \subset L \subset G$ of a finite group G, and assume that $W = \operatorname{Res}_H^L(\widetilde{W})$ for some finite-dimensional $k(L)$-module $\widetilde{W}$. Then*

(i) *The group $\Gamma := N_L(H)/H$ acts on $U := \operatorname{Ind}_H^G(W)$ via*

$$(\gamma \cdot f)(t_g) := \gamma(f(t_{g\gamma}))$$

for $\gamma \in N_L(H)$.

(ii) *For $\widetilde{W} \neq 0$ this action is faithful, that is, it injects $\Gamma \hookrightarrow \operatorname{Aut}_{k(G)}(U)$.*

(iii) *The isomorphism*

$$(V \otimes_k \operatorname{Ind}_H^G(W))^G \cong (\operatorname{Res}_H^G(V) \otimes_k W)^H.$$

of Proposition 3.6.1 is also a $k(\Gamma)$-isomorphism, assuming one lets $k(\Gamma)$ act as follows:

- *on the left, solely in the factor $\operatorname{Ind}_H^G(W)$, while*
- *on the right, diagonally in both factors of $\operatorname{Res}_H^G(V) \otimes_k W$.*

Proof. It is straightforward to check that the definition in (i) describes an action of $N_L(H)$ on $\operatorname{Ind}_H^G(W)$ with H acting trivially, and that this action commutes with the $k(G)$-module structure.

To see that $\widetilde{W} \neq 0$ implies the action is faithful, pick any $\tilde{w} \neq 0$ in $\widetilde{W}$ and consider the element f of $\operatorname{Ind}_H^G(W)$ defined by

$$f(t_g) = \begin{cases} g^{-1}(w) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

An element γ in $N_L(H)$ sends f to the function

$$(\gamma \cdot f)(t_g) = \begin{cases} \gamma((g\gamma)^{-1}(w)) = g^{-1}(w) & \text{if } g\gamma \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\gamma \cdot f = f$ if and only if γ lies in H.

It is also not hard to check that the isomorphism given in the proof of Proposition 3.6.1 is $k(\Gamma)$-equivariant for the actions described. □

We next provide the proof of our main application, Corollary 1.2.2, resolving in the affirmative both Conjecture 3 and Question 4 in [20].

Proof of Corollary 1.2.2. Recall that the ground field k is arbitrary and $H \subset G$ are two nested finite subgroups of $GL(V)$ where V is a finite dimensional k-vector space. We have assumed that the ring of invariants $k[V]^G$ is a polynomial algebra, so it is its own Noether normalization and may be taken as R in the results established in this section. The group $C = \langle c \rangle$ is a cyclic subgroup generated by a regular element c in G, with eigenvalue ω on some regular eigenvector in V. We set $\Gamma := N_G(H)/H$ and have given $k[V]^H$ the $k(\Gamma \times C)$-structure in which c scales the variables in V^* by ω, and Γ acts by linear substitutions.

What we must prove is that in $\mathbf{R}(k(\Gamma \times C))$

$$\sum_{i \geq 0} (-1)^i \left[\mathrm{Tor}^{k[V]^G}(k[V]^H, k) \right] = [k(H \backslash G)] \,. \tag{3.6.1}$$

So ignoring the action of Γ this will imply that

$$X(t) = \frac{\left[k[V]^H\right](t)}{\left[k[V]^G\right](t)} \in \mathbb{Z}[[t]],$$

is a polynomial in t, and the action of C on the set $X = H\backslash G$ then gives a triple $(X, X(t), C)$ that exhibits the cyclic sieving phenomenon of [21]. Specifically, for any element c^j in C, the cardinality of its fixed point subset $X^{c^j} \subset X$ is given by evaluating $X(t)$ at a complex root-of-unity $\hat{\omega}^j$ of the same multiplicative order as c^j, viz., $|X^{c^j}| = [X(t)]_{t=\hat{\omega}^j}$.

In this context, we apply Theorem 1.2.1, together with the results of this section, taking $\widetilde{W} = k$ to be the trivial $k(G)$-module. Then $W := \mathrm{Res}^G_H(\widetilde{W}) = k$ is the trivial $k(H)$-module, and

$$U := \mathrm{Ind}_H^G(W) = \mathrm{Ind}_H^G k = k(H\backslash G)$$

where $k(H\backslash G)$ carries the $k(\Gamma \times C)$-module structure as described in the statement of Corollary 1.2.2. So we have

$$M = (U \otimes_k k[V])^G \cong (k \otimes_k k[V])^H = k[V]^H,$$

and unraveling the notations, Theorem 1.2.1 tells us that equality (3.6.1) holds in $\mathbf{R}(k(\Gamma \times C))$ as required. If we ignore the Γ-action and compare

Brauer character values for each element c^{-j} in C on the two sides of equality (3.6.1) we find, on the left side

$$\sum_{i\geq 0}(-1)^i \left[\mathrm{Tor}^{k[V]^G}(k[V]^H,k)\right](t)\Bigg|_{t=\hat{\omega}^j} = \frac{\left[k[V]^H\right](t)}{\left[k[V]^G\right](t)}\Bigg|_{t=\hat{\omega}^j} = X(\hat{\omega}^j),$$

where $\hat{\omega}$ is the Brauer lift to $\mathbb{C}^\times$ of $\omega \in k^\times$, while on the right it is the number of fixed points for c^j permuting the elements of the set $X = H\backslash G$. □

4. Character Values of the Binary Icosahedral Group

We note here an interesting corollary of Theorem 1.3.1, relating the Brauer character value $\chi_U(g)$ of a regular element g to evaluations of the Hilbert series for the module of relative invariants $(U \otimes_k k[V])^G$ at the lift of a regular eigenvalue for g. We then apply this to the binary octahedral group.

4.1. **A General Character Value Corollary.** Given a graded algebra R and graded R-module M, as in Section 2, define

$$X_{M,R}(t) = \frac{[M](t)}{[R](t)} = \sum_{i\geq 0}(-1)^i \left[\mathrm{Tor}_i^R(M,k)\right](t).$$

where the second equality comes from Proposition 2.3.1(iv).

Proposition 4.1.1. *Let G be a finite subgroup of $GL(V)$, and g a regular element of G with regular eigenvector v and associated eigenvalue ω. Let U be a $k(G)$-module and denote its Brauer character by $\chi_U : G \to \mathbb{C}$. Let $\hat{\omega} \in \mathbb{C}^\times$ be the Brauer lift of ω.*

Assume there exists a graded Noether normalization $R \subset k[V]^G$ having the following properties.[2]

(i) *G acts freely on the fiber $\Phi_v = \phi^{-1}(\phi(v))$ for the map $\phi : V \to \mathrm{Spec}(R)$.*
(ii) *For every G-orbit Gw_i in Φ_v that has $\omega Gw_i = Gw_i$, the unique element $g_{\omega,w_i} \in G$ for which $\omega w = g_{\omega,w_i} w$ has the same Brauer character value $\chi_U(g_{\omega,w_i}) = \chi_U(g)$.*
(iii) *$X_{k[V]^G,R}(\hat{\omega}) \neq 0$.*

Then

$$\chi_U(g) = X_{M,k[V]^G}(\hat{\omega})$$

where $M := (U \otimes_k k[V])^G$.

Proof. First note that Proposition 2.3.1(iv) implies

$$X_{M,R}(t) = X_{M,k[V]^G}(t) \cdot X_{k[V]^G,R}(t)$$

[2]These hypotheses would hold if $k[V]^G$ were polynomial and one took $R = k[V]^G$, but we *do not* assume this.

and hence

$$X_{M,R}(\hat\omega) = X_{M,k[V]^G}(\hat\omega) \cdot X_{k[V]^G,R}(\hat\omega). \tag{4.1.1}$$

Consider the cyclic group $C = \langle \tau \rangle \subset \mathrm{Aut}_{k(G)} V$ whose generator τ acts as the scalar ω^{-1} on V, as in the proof of Theorem 1.2.1. Then τ scales V^* by ω, and acts on the graded rings and modules

$$R = k[V]^G,\ k[V],\ M := (U \otimes k[V])^G$$

by the scalar ω^j on their j^{th} homogeneous component. Hypothesis (i) allows us to apply Theorem 1.3.1. If we consider only the $k(C)$-module structure then in $\mathbf{R}(k(C))$

$$[\mathrm{Tor}^R(M,k)] = \left[(U \otimes_k A(\Phi_v))^G\right].$$

Therefore

$$\begin{aligned} X_{M,R}(\hat\omega) &= \chi_{\mathrm{Tor}^R(M,k)}(c) \\ &= \chi_{(U\otimes_k A(\Phi_v))^G}(c) \\ &= \sum_{i:\omega G w_i = G w_i} \chi_{B_{Gw_i}}(c) \cdot \chi_U(g^{-1}_{c,w_i}) \qquad (4.1.2)\\ &= \chi_U(g^{-1}) \cdot \sum_{i:\omega G w_i = G w_i} \chi_{B_{Gw_i}}(c) \end{aligned}$$

where the first equality uses the definition of the scalar action on R, M, $\mathrm{Tor}^R(M,k)$, the second equality is equation (3.5.2) of Lemma 3.5.1 and the last equality used hypothesis (ii) above.

In particular, taking $U = k$ the trivial $k(G)$-module in (4.1.2) gives

$$X_{k[V]^G,R}(\hat\omega) = \sum_{i:\omega G w_i = G w_i} \chi_{B_{Gw_i}}(c). \tag{4.1.3}$$

Putting together equations (4.1.1), (4.1.2), and (4.1.3), one obtains

$$X_{M,k[V]^G}(\hat\omega) \cdot X_{k[V]^G,R}(\hat\omega) = X_{M,R}(\hat\omega) = \chi_U(g) \cdot X_{k[V]^G,R}(\hat\omega) \tag{4.1.4}$$

Since by hypothesis (iii) $X_{k[V]^G,R}(\hat\omega)$ is a nonzero factor of the extreme left and right terms of this string of equalities, we may divide by it yielding the desired equality. □

4.2. **The Binary Icosahedral Group.** We give here an application of Proposition 4.1.1 that highlights the flexibility of using more than one Noether normalization $R \subset k[V]^G$.

The finite subgroups G inside $SL(2,\mathbb{C})$ are called the **binary polyhedral groups**; see Springer [29, §4.4] for a discussion, including their classification. These groups consist of:

- two infinite families, the binary cyclic and dihedral groups, and
- three exceptional groups, the binary tetrahedral, octahedral, and icosahedral groups.

The binary icosahedral group is the largest of the exceptions, having 120 elements.

Proposition 4.2.1. *Let G be the binary icosahedral group, and g any element of G with eigenvalues ω, ω^{-1}, Then for any $\mathbb{C}(G)$-module U one has*

$$\chi_U(g) = X_{M,\mathbb{C}[V]^G}(\omega)$$

where $V = \mathbb{C}^2$ and $M := (U \otimes_{\mathbb{C}} \mathbb{C}[V])^G$.

Proof. We wish to apply Proposition 4.1.1 with v an ω-eigenvector for g. To do so, note that for any finite subgroup G of $SL(2,\mathbb{C})$, any nonzero vector $w \in \mathbb{C}^2$ always has a regular G-orbit Gw because, if $g \in G$ fixes w, then both its eigenvalues will be 1, and g being semisimple must be the identity. Thus any choice of a graded Noether normalization $R \subset \mathbb{C}[V]^G$ has G acting freely on the fiber Φ_v, and hypothesis (i) of Proposition 4.1.1 is satisfied.

Springer [29, §4.5] computes a presentation for the G-invariants, namely

$$\mathbb{C}[V]^G = \mathbb{C}[f_{12}, f_{20}, f_{30}]/(f_{12}^5 + f_{20}^3 + f_{30}^2)$$

where f_d is a homogeneous generator of degree d, and hence

$$\left[\mathbb{C}[V]^G\right](t) = \frac{1-t^{60}}{(1-t^{12})(1-t^{20})(1-t^{30})}.$$

The elements $g \in G$ have orders $1, 2, 3, 4, 5, 6, 10$, and there are unique conjugacy classes of elements with each of the orders $1, 2, 3, 4, 6$. To verify the remaining hypotheses (ii) and (iii) in Proposition 4.1.1 for each element of G, we use two different Noether normalizations $R \subset \mathbb{C}[V]^G$, the choice depending on the order of g.

For elements g of order dividing 20 (that is, orders $1, 2, 4, 5, 10$), we use

$$R = \mathbb{C}[f_{12}, f_{30}]$$
$$X_{\mathbb{C}[V]^G,R}(t) = \frac{1-t^{60}}{1-t^{20}} = 1 + t^{20} + t^{40}$$

Note that $X_{\mathbb{C}[V]^G,R}(\hat{\omega}) = 3 \neq 0$ for such elements g, so hypothesis (iii) of Proposition 4.1.1 will be satisfied. Springer's discussion (loc. cit.) of the above presentation $\mathbb{C}[V]^G$ shows that the ring R is in fact the invariant ring $\mathbb{C}[V]^\Gamma$ for the subgroup $\Gamma = \langle G, \zeta \rangle$ of $GL(2,\mathbb{C})$, where ζ is the scalar matrix for a primitive cube root-of-unity. Since ζ commutes with ω and G, this means that the fiber Φ_v either consists of

- the single G-orbit Gv, having $g_{\omega,v} = g$, or
- three orbits $Gv, G\zeta v, G\zeta^2 v$, in which case all three correspond to the same element $g_{\omega,\zeta^i v}$ whose existence is asserted in Proposition 4.1.1 (ii), viz.,

  $$g = g_{\omega,v} = g_{\omega,\zeta v} = g_{\omega,\zeta^2 v},$$

  if one uses the three orbit representatives $v, \zeta v, \zeta^2 v$.

So in these cases hypothesis (ii) of Proposition 4.1.1 is satisfied.

For elements of orders $3, 6$, we use

$$R = \mathbb{C}[f_{12}, f_{20}]$$
$$X_{\mathbb{C}[V]^G,R}(t) = \frac{1-t^{60}}{1-t^{30}} = 1 + t^{30}$$

Note that $X_{\mathbb{C}[V]^G,R}(\hat{\omega}) = 2 \neq 0$ for such elements g, so hypothesis (iii) of Proposition 4.1.1 will be satisfied. On the other hand, hypothesis (ii) will always be satisfied in this case, because all the elements g_{ω,w_i} for the various orbits Gw_i in Φ_v have the same order as ω, namely 3 or 6. So they will all be conjugate since there are unique conjugacy classes of elements of these orders. □

Remark 4.2.2.
One might be tempted to apply the same method used to prove Proposition 4.2.1 for finite subgroups G of $SL(2, k)$ with other fields k, even in positive characteristic. However, in order to check the hypothesis (ii) of Proposition 4.1.1, one needs to be careful that the ω-eigenvector v for g and the Noether normalization R avoid having any vector w in the fiber Φ_v which is fixed by a *transvection* [25, §8.2] of G.

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Département de mathémathiques et de statistique, Université de Montréal, C.P. 6128, succursale Centre-ville, Montréal (Québec), Canada H3C 3J7
E-mail address: broera@DMS.UMontreal.CA

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: reiner@math.umn.edu

AG-Invariantentheorie, Mittelweg 3, D 37133 Friedland, Federal Republic of Germany
E-mail address: larry_smith@GMX.net

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: webb@math.umn.edu