Proceedings of Symposia in Pure Mathematics
Volume 47 (1987)

Subgroup Complexes

P. J. WEBB

1. Introduction. We survey the general properties of the simplicial complex of p-subgroups of a finite group G. The partially ordered set of p-subgroups of G together with the conjugation action on this poset carries the information inherent in the p-local structure of G. By forming a simplicial complex out of this poset and considering the group action on it we have a systematic way of dealing with global features of the p-local structure.

The p-subgroups complex is really a geometry for G. Whatever one means by a geometry, there is usually a simplicial complex involved, it is associated to a prime p, and the stabilizers of simplices are treated as analogues of parabolic subgroups. One can take the view that the most canonically defined nontrivial simplicial complex on which G acts, associated to the prime p, is the p-subgroups complex. In all situations where buildings are defined, the p-subgroups complex coincides, in a certain sense, with the building. It is one of the themes of this survey that the p-subgroups complex provides an analogue defined for every finite group and every prime of a building.

Group actions on suitable spaces have long been used to elucidate the group structure, especially with certain classes of infinite groups such as arithmetic groups. The philosophy is to obtain the group structure in terms of the stabilizer subgroups. The action on the p -subgroups complex is eminently suitable for this application and we show how to use the structure of this complex to determine the cohomology of a finite group. This situation with a group action is one where the equivariant cohomology spectral sequence is also used. The approach described here yields stronger results for finite groups and it can be fed back in such a way that the structure of the spectral sequence is made transparent, to the extent that this spectral sequence becomes redundant. It remains to be seen whether a similar approach free of equivariant cohomology can be made to work for infinite groups.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20J06; Secondary 20D15, 20D25,20D30.

We see that the subject of this article enjoys the following close relationships with other areas of group theory:

geometries cohomology

subgroup
complexes

p-local structure

In §2 we describe the p-subgroups complex and also the complexes of elementary abelian p-subgroups, subgroups $H = O_p(N_G(H))$, and normal chains of p-subgroups, which are homotopy equivalent to it. In §3 we discuss the homological structure of the p-subgroups complex and the application to group cohomology. §4 is about Euler characteristics, and in §§5 and 6 we consider a general definition of a Steinberg module. According to this definition, every finite group has a Steinberg module at every prime. §7 is an appendix containing lemmas about chain complexes that are sometimes useful. In the early part of this survey we omit most proofs because the results can almost all be found written up in one or other of the cited references. In §§6 and 7 this is not the case, and we supply proofs.

2. The complexes

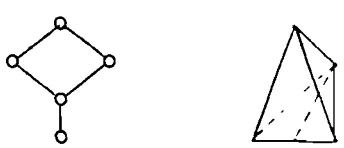
2.1. *Construction.* Given a poset $\mathcal{P}$ we form a simplicial complex Δ where the set of n-simplices of Δ is

$$\{x_0 < x_1 < \cdots < x_n \mid x_i \in \mathcal{P}\}.$$

These are chains of elements of $\mathcal{P}$ of length $n+1$, where we require strict inequalities in the chain. The faces of such a simplex are the shorter subchains. If G acts on $\mathcal{P}$ as a group of poset automorphisms then G will act simplicially on Δ.

Example:

Poset Simplicial Complex



The way the chains in the poset overlap is the way the simplices are glued together.

2.2. *The posets.* Fix a prime number p; we define

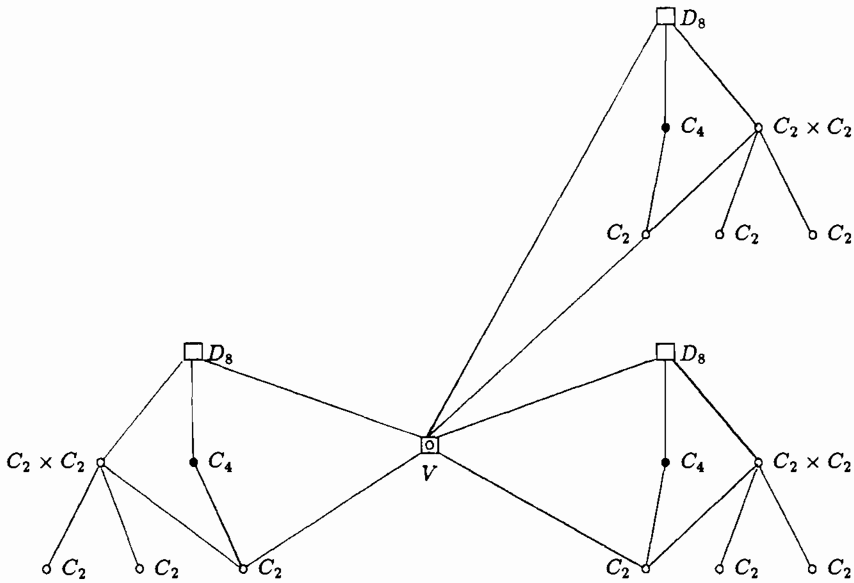
$$\mathcal{S}_p(G) = \{\text{all nonidentity } p\text{-subgroups of } G\},$$

$$\mathcal{A}_p(G) = \{\text{nonidentity elementary abelian } p\text{-subgroups of } G\},$$

$$\mathcal{B}_p(G) = \{H \leq G \mid H = O_pN_G(H), H \neq 1\}.$$

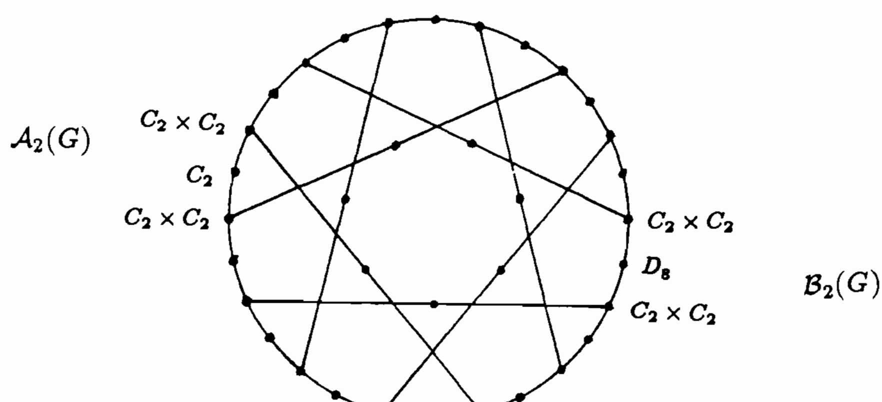
In this generality the poset $\mathcal{B}_p(G)$ seems to have been first considered by S. Bouc. If H is a subgroup satisfying the condition $H = O_pN_G(H)$ of the last definition, Alperin calls $N_G(H)$ a *parabolic subgroup*. When G is a finite Chevalley group in defining characteristic p, this agrees with the usual definition and the subgroups in $\mathcal{B}_p(G)$ are precisely the unipotent radicals of the parabolic subgroups.

EXAMPLES. 1. $G = S_4, p = 2$.



This is the whole of $\mathcal{S}_2(S_4)$. The three wings should be the same, although one of them is drawn above the other two. The subgroups in $\mathcal{A}_2(S_4)$ are shown as open circles $\circ$, and those in $\mathcal{B}_2(S_4)$ are squares $\square$. Although these are evidently different posets, they are all contractible; indeed $\mathcal{A}_2(S_4)$ and $\mathcal{B}_2(S_4)$ are trees.

2. $G = \mathrm{GL}(3,2)$, $p = 2$. $\mathcal{A}_2(G)$ and $\mathcal{B}_2(G)$ are both isomorphic to the barycentric subdivision of the incidence graph of the projective plane of order 2, but with different subgroups of G as the vertices.



We describe another simplicial complex, which was first considered by G. R. Robinson in his reformulation of Alperin's conjecture. This is denoted $\mathcal{R}_p(G)$ and has as its n-simplices the chains $P_0 < P_1 < \cdots < P_n$ of nonidentity p-subgroups of G such that $P_i \lhd P_n$ for all i. The vertices are thus the same as the vertices in the simplicial complex associated to $\mathcal{S}_p(G)$, but there are fewer higher-dimensional simplices. Call a chain of subgroups in which all subgroups are normal in the largest a *normal chain.* We may also consider the complex of normal chains of subgroups in $\mathcal{B}_p(G)$, which we will not bother to name. This complex may also be included in the list in the following theorem.

2.3. THEOREM (QUILLEN [7], BOUC [2], THÉVENAZ, WEBB). *$\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{R}_p(G)$ are all G-homotopy equivalent.*

The effect of this theorem is that for many purposes it does not matter which of the simplicial complexes we use, but before discussing this let's look at the definitions.

G-Homotopy equivalence. Spaces X_1 and X_2 with G-actions are said to be G-homotopy equivalent if there exist G-equivariant maps $f : X_1 \to X_2$ and $g : X_2 \to X_1$ with $fg \simeq 1$ and $gf \simeq 1$ given by G-equivariant homotopies. Passing to chain complexes, this would imply that the chain complexes of X_1 and X_2 are chain homotopy equivalent in such a way that all of the maps used to establish the chain homotopy equivalence (including the two maps of degree $+1$) are G-homomorphisms. This evidently implies that the homology groups of the spaces are isomorphic as G-modules. It also implies that X_1/G and X_2/G are homotopy equivalent. The following theorem is fundamental:

2.4. THEOREM (BREDON [3, §II]). *Let X_1 and X_2 be G-CW-complexes. A G-equivariant map $f : X_1 \to X_2$ is a G-homotopy equivalence if and only if $f|_{X_1^H} : X_1^H \to X_2^H$ is a homotopy equivalence for all subgroups $H \leq G$.*

The equivalent condition here is what is in fact used to prove the G-homotopy equivalence of the four posets. Putting the information together we obtain:

COROLLARY. *In each dimension the homology groups of the four posets $\mathcal{S}_p(G), \mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{R}_p(G)$ are isomorphic as G-modules. For any subgroup H, the fixed point sets under the action of H are homotopy equivalent.*

2.5. *First properties of the complexes.* We state them just for $\mathcal{S}_p(G)$, and by the above they evidently hold for all the complexes.

1. $\mathcal{S}_p(G_1 \times G_2) \simeq \mathcal{S}_p(G_1) * \mathcal{S}_p(G_2)$ (the topological join) (Quillen [7]).
2. For a finite Chevalley group in defining characteristic p, $\mathcal{B}_p(G)$ is the building of G. This is because as a poset it consists of the unipotent radicals of the (nontrivial) parabolic subgroups of G and the inclusion relationship between parabolics is exactly reversed on taking unipotent radicals, so we are looking at the opposite of the poset which normally defines the building. But a poset and its opposite give the same simplicial complex. The force of this remark is that $\mathcal{B}_p(G)$ is a generalization to all finite groups of the building of a Chevalley group.

3. If $O_p(G) \neq 1$ then $\mathcal{S}_p(G)$ is contractible. This is well illustrated in the case of S_4 at $p = 2$. It is appropriate to mention

QUILLEN'S CONJECTURE. *If $\mathcal{S}_p(G)$ is contractible then $O_p(G) \neq 1$.*

Quillen showed in [7] that this is true for soluble groups and groups of p-rank 2. It has recently been shown by Thévenaz and Lyons using the classification of finite simple groups that Quillen's conjecture is also true for these groups. This means that if G is a finite simple group, then $\mathcal{S}_p(G)$ is not contractible.

3. The structure theorem and cohomology. Let $\mathbf{Z}_p$ denote the p-adic integers. If

$$D. = \cdots D_{n+1} \xrightarrow{d_{n+1}} D_n \xrightarrow{d_n} D_{n-1} \cdots$$

is a chain complex of $\mathbf{Z}_pG$-modules, we say it is *acyclic split* if it has zero homology and for every n the sequence $0 \to \ker d_n \to D_n \to \operatorname{Im} d_n \to 0$ is split. If $D.$ is augmented by an epimorphism $D_0 \to \mathbf{Z}_p$, by saying $D.$ is *acyclic split augmented* we mean the same condition applied to the augmented complex (i.e., zero reduced homology, all the sequences split, and the augmentation splits).

3.1. STRUCTURE THEOREM (WEBB [10]). *Let Δ be the simplicial complex of one of $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, or $\mathcal{R}_p(G)$. Let $C.(\Delta) = C_d(\Delta) \to C_{d-1}(\Delta) \to \cdots \to C_0(\Delta)$ be the chain complex of Δ over $\mathbf{Z}_p$. Then $C.(\Delta)$ has an acyclic split augmented subcomplex $D.$ so that $C.(\Delta)/D.$ is a complex of projective modules.*

This means $C_r(\Delta) = D_r \oplus P_r$ for all r where P_r is projective, and it also follows from the long exact sequence in homology that $C.(\Delta)$ and $C.(\Delta)/D.$ have the same reduced homology. I propose the following:

3.2. CONJECTURE. *The reduced homology groups of $\mathcal{S}_p(G)$, when completed at p, are all projective $\mathbf{Z}_pG$-modules.*

The main evidence for this is the above theorem, which gives the homology as the homology of a complex of projective modules. It is an easy deduction from this theorem that the conjecture holds if $\mathcal{S}_p(G)$ has only one nonzero reduced homology group (as happens, in particular, in the case of a building). It is also the case that for all finite groups G, $\tilde{H}_0(\mathcal{S}_p(G))$ is projective at the prime p, because when this reduced homology is nonzero, G has a strongly p-embedded subgroup H so that $\mathbf{Z}_p\uparrow_H^G \cong \mathbf{Z}_p \oplus \tilde{H}_0(\mathcal{S}_p(G))_p$ (see [7]). The nontrivial summand is always projective, and the situation is rather well controlled. Presumably if the conjecture were true, the projectivity in higher dimensions might correspond to other structural statements about G of a similar nature.

Again let Δ be the simplicial complex of one of the above four posets and let Δ_r denote the set of simplices in dimension r. The structure of the chain complex of Δ has the following consequence.

3.3. THEOREM (WEBB [10]). *For all integers n and for all $\mathbf{Z}_pG$-modules M there are split exact sequences*

$$0 \to \hat{H}^n(G,M)_p \to \bigoplus_{\sigma\in\Delta_0/G} \hat{H}^n(G_\sigma,M)_p \xrightarrow{(\phi_{\sigma\tau})} \bigoplus_{\tau\in\Delta_1/G} \hat{H}^n(G_\tau,M)_p \to \cdots \to \bigoplus_{\sigma\in\Delta_d/G} \hat{H}^n(G_\sigma,M)_p \to 0$$

and

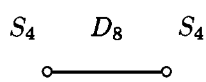
$$0 \leftarrow \hat{H}^n(G,M)_p \leftarrow \bigoplus_{\sigma\in\Delta_0/G} \hat{H}^n(G_\sigma,M)_p \xleftarrow{(\psi_{\sigma\tau})} \bigoplus_{\tau\in\Delta_1/G} \hat{H}^n(G_\tau,M)_p \leftarrow \cdots \leftarrow \bigoplus_{\sigma\in\Delta_d/G} \hat{H}^n(G_\sigma,M)_p \leftarrow 0,$$

where if σ is a face of τg then

$$\phi_{\sigma\tau} = (-1)^{(\sigma|\tau g)} C_{g^{-1}} \circ \operatorname{res}^{G_\sigma}_{G_{\tau g}}, \qquad \psi_{\sigma\tau} = (-1)^{(\sigma|\tau g)} \operatorname{cores}^{G_\sigma}_{G_{\tau g}} \circ c_g,$$

and otherwise $\phi_{\sigma\tau}$ are zero. Here $(\sigma \mid \tau g)$ denotes the orientation of the embedding of σ in τg.

We will illustrate the theorem in the case $G = \mathrm{GL}(3,2)$, $p = 2$. Let Δ be the building of G. Representatives of the simplex stabilizers are given in the following picture of Δ/G:



Preciously in §2 we saw that $\mathcal{B}_2(G)$ gives the barycentric subdivision of this complex Δ, but evidently the two are G-homotopy equivalent and the conclusions of Theorem 3.1 are independent of G-homotopy equivalence. The first sequence in Theorem 3.3 is

$$0 \to \hat{H}^n(G,M)_2 \xrightarrow{(-\operatorname{res},\operatorname{res})} \hat{H}^n(S_4,M)_2 \oplus \hat{H}^n(S_4,M)_2 \xrightarrow{\binom{\operatorname{res}}{\operatorname{res}}} \hat{H}^n(D_8,M)_2 \to 0,$$

and by the theorem this is split exact. For example, in the case of $H_1(G,\mathbf{Z}) = \hat{H}^{-2}(G,\mathbf{Z}) \simeq G/G'$ it is

$$1 \to 1 \to C_2 \times C_2 \to C_2 \times C_2 \to 1.$$

Here the restriction homomorphisms are the classical transfer. For the Schur multiplier, $H_2(G,\mathbf{Z}) = \hat{H}^{-3}(G,\mathbf{Z})$, the sequence is

$$1 \to C_2 \to C_2 \times C_2 \to C_2 \to 1.$$

These are the right answers!

Computationally, we would obtain the cohomology of G if we knew all the right-hand terms in these sequences. If we are only interested in isomorphism types of individual groups we do not need to know exactly what the maps are, because of the splitting. These maps become important when we are interested in cohomology ring structure, and then the left-hand morphism from $\hat{H}^n(G, M)_p$ is a ring homomorphism.

You may have noticed in Theorem 3.3 that the restriction sequences with the $\hat{H}^n(G, M)_p$ terms removed constitute the E_1 page of the equivariant cohomology spectral sequence for the action of G on Δ ([**4**, VII.8.1]). The exactness of the sequences implies that the E_2 page only has nonzero terms on the fibre, and the spectral sequence stops there.

PROOF OF 3.3. $\mathrm{Ext}^n_{\mathbf{Z}_pG}(\tilde{C}.(\Delta), M_p)$ is the first sequence (for positive values of n), where $\tilde{C}.(\Delta)$ is the augmented chain complex of Δ, and the second sequence is $\mathrm{Ext}^n_{\mathbf{Z}_pG}(\tilde{C}.(\Delta)^*, M_p)$, where * denotes the dual. This identification is almost the same as [**4**, p. 173]. We apply the structure Theorem 3.3. The Ext groups vanish on the projective modules, and what is left is Ext of a split acyclic complex. This is again split acyclic.

4. Euler characteristics. We take Δ to be the simplicial complex of one of $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, or $\mathcal{R}_p(G)$.

4.1. THEOREM (WEBB [**10**]). *Δ/G is* $\mathrm{mod}\, p$ *acyclic.*

This means the chain complex of Δ/G reduced $\mathrm{mod}\, p$ has the homology of a point. In fact, when G has p-rank 2 this result forces Δ/G to be contractible. The condition p-rank 2 means that the largest elementary abelian p-subgroup is $C_p \times C_p$ and in this case $\mathcal{A}_p(G)$ is a graph. Thus Δ/G is homotopy equivalent to a graph (no matter which simplicial complex Δ is) and it is also connected (because of conjugacy of Sylow p-subgroups). One readily sees that a connected graph which is $\mathrm{mod}\, p$ acyclic is a tree, because the homology of a graph is torsion free of rank depending on the number of cycles in the graph. For example, $\mathcal{A}_2(S_5)/S_5$ is the tree

$$\begin{array}{cccc} C_2 & C_2 \times C_2 & C_2 & C_2 \times C_2 \\ \circ & \circ & \circ & \circ \end{array}$$

where the vertices are the two conjugacy classes of C_2 subgroups and the two conjugacy classes of $C_2 \times C_2$ subgroups. This quotient complex is some kind of global picture of the p-local structure of the group, and it seems very strange that in general (when G has p-rank 2) it should be a tree.

4.2. CONJECTURE. *If Δ is the simplicial complex of $\mathcal{S}_p(G)$ then Δ/G is contractible.*

If this were true it would be a generalization of what happens for buildings, where Δ/G is a single simplex.

The next result has been proved in many ways since the original proof by K. S. Brown. There is now a more general result of this kind due to Brown and Thévenaz [**11**] which subsumes a lot of earlier work. Special cases of [**11**] have been obtained independently by Hawkes, Isaacs, and Ozaydin.

4.3. COROLLARY (K. S. BROWN). $\chi(\mathcal{S}_p(G)) \equiv 1(\mathrm{mod}\, |G|_p)$.

PROOF. From the structure theorem, $C_r(\Delta) = D_r \oplus P_r$ for all r where P_r is a projective $\mathbf{Z}_pG$-module. Thus $\chi(\mathcal{S}_p(G)) = \sum_{r=0}^{d}(-1)^r \operatorname{rank} C_r(\Delta) = \sum_{r=0}^{d}(-1)^r \operatorname{rank} D_r + \sum_{r=0}^{d}(-1)^r \operatorname{rank} P_r$. The first number is 1 because $D.$ has the homology of a point, and the second is congruent to 0 (mod $|G|_p$) because all the modules are projective.

5. The Steinberg module. Let $a(G) = K_0(\mathrm{Latt}_{\mathbf{Z}_pG}, 0)$ be the Green ring of $\mathbf{Z}_pG$-lattices. This is the free abelian group with the set of isomorphism classes of indecomposable $\mathbf{Z}_pG$-lattices as a basis. Because the Krull-Schmidt theorem holds, knowing a lattice as an element of $a(G)$ is equivalent to knowing its isomorphism type. As before, let Δ be the simplicial complex of one of $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, or $\mathcal{R}_p(G)$ and let $C.(\Delta)$ be its chain complex over $\mathbf{Z}_p$. We will see in §6 that it does not matter which simplicial complex we take.

DEFINITION. The Steinberg module of G at the prime p is

$$\mathrm{St}_p(G) = -\mathbf{Z}_p + \sum_{r=0}^{d}(-1)^r C_r(\Delta),$$

the sum taken in $a(G)$.

A similar notation is used by Bouc [**1**] for the corresponding element of the ring of Brauer characters. It may be misleading to call it a module, because by this definition it is only a virtual module. It is intriguing that in all known cases, $\mathrm{St}_p(G)$ is $\pm$ an actual module.

5.1. THEOREM. *$\mathrm{St}_p(G)$ is a virtual projective; that is, there exist projective $\mathbf{Z}_pG$-modules P and Q so that $\mathrm{St}_p(G) = P - Q$ in $a(G)$.*

PROOF. By the structure theorem $C_r(\Delta) = D_r \oplus P_r$ where $D.$ is an acyclic split augmented subcomplex and each P_r is projective. All of the indecomposable summands of the D_r cancel out on taking alternating sums, because of the splitting. Hence $\mathrm{St}_p(G) = \sum_{r=0}^{d}(-1)^r P_r$ is a virtual projective.

5.2. *Properties of* $\mathrm{St}_p(G)$. 1. If G is a finite Chevalley group then $\mathrm{St}_p(G) = (-1)^{\operatorname{rank}(G)-1}\,\mathrm{St}$, where St is the usual Steinberg module and $\operatorname{rank}(G)$ is the rank of the root system. This follows from the Solomon-Tits theorem, which tells us that the top homology of $\mathcal{B}_p(G)$ is St and all other reduced homology is zero. Since we know in advance that St is projective over $\mathbf{Z}_pG$, it follows that it splits off from the chain complex, and in fact using the structure theorem the whole chain complex is the splice of split short exact sequences of $\mathbf{Z}_pG$-lattices. This was also observed by Kuhn and Mitchell [**6**]. The discrepancy with the minus sign between $\mathrm{St}_p(G)$ and St arises because $\mathcal{B}_p(G)$ has dimension $\operatorname{rank}(G) - 1$.

2. $\mathrm{St}_p(G_1 \times G_2) = -\mathrm{St}_p(G_1) \otimes \mathrm{St}_p(G_2)$. This follows from $\mathcal{S}_p(G_1 \times G_2) \simeq \mathcal{S}_p(G_1) * \mathcal{S}_p(G_2)$.

3. Unlike the usual Steinberg character, $\mathrm{St}_p(G)$ is in general not simple. It need not even be indecomposable.

4. If $O_p(G) \neq 1$ then $\mathrm{St}_p(G) = 0$. In particular, p-groups have trivial Steinberg module. If $|G|$ is prime to p then $\mathrm{St}_p(G) = -\mathbf{Z}_p$.

5. $\mathrm{St}_p(G)$ is self-dual.

6. The simple modules whose projective covers appear in the support of $\mathrm{St}_p(G)$ all have dimension at least p. In particular, P_1 never appears in the support of $\mathrm{St}_p(G)$. This follows from the following proposition, which was also used to compute the ensuing table.

5.3. PROPOSITION. *Assume $p \| |G|$. For each simple $\mathbf{Z}_pG$-module S, the multiplicity of the projective cover P_S in* $\mathrm{St}_p(G)$ *is*

$$\sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma}(\text{multiplicity of } P_1 \text{ for } G_\sigma \text{ in } S|_{G_\sigma}).$$

We may take Δ to be any of the four simplicial complexes described in §2. In the parenthetic term above we compute how many times the projective cover over $\mathbf{Z}_pG_\sigma$ of the trivial module for $\mathbf{Z}_pZG_\sigma$ is a summand of $S|_{G_\sigma}$. This term is zero if $\dim S < |G_\sigma|_p$, since $\dim P_1 \geq |G_\sigma|_p$. When Δ is the simplicial complex of any of the four posets $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{R}_p(G)$ we always have $|G_\sigma|_p \geq p$, hence the assertion in 6 that P_S does not occur at all if $\dim S < p$.

Steinberg modules at $p = 2$:

S_3	$P_{(2,1)}$
S_4	0
S_5	$-P_{(3,2)}$
S_6	$-P_{(3,2,1)}$
S_7	$P_{(4,3)} + P_{(4,2,1)}$
A_7	$2P_{20} + P_{14}$
M_{11}	$-3P_{44} - 2P_{16} - 2P_{\overline{16}}$
M_{12}	$10P_{144}$
M_{24}	P_{1792}

For the symmetric groups the notation P_λ denotes the projective cover of the modular irreducible corresponding to the p-regular partition λ. Otherwise, irreducibles are referred to by their dimensions. I am indebted to L. G. Griffiths, S. E. Rees, M. A. Ronan, and S. Smith for the more difficult calculations.

6. Steinberg module inversion. This is a process like Möbius inversion, which is familiar to workers in several areas. In the context of representations of Chevalley groups it takes the form of a duality between the Steinberg module and the trivial module. We take an approach which was first used in a less general context by Bouc [**1**], and go on to deduce an application to group cohomology, and also Robinson's reformulation of Alperin's conjecture. First the statement of the inversion formula in the language of the previous sections (cf. Proposition 4 of [**1**], but note that Bouc works in the ring of Brauer characters).

6.1. THEOREM. *In $a(G)$,*

$$\mathbf{Z}_p = - \sum_{\{1 \leq P \leq G \mid P \text{a } p\text{-subgroup}\}/G} \mathrm{St}_p(N_G(P)/P) \uparrow_{N_G(P)}^G .$$

The vertical arrow means induction. In this formula we regard $\mathrm{St}_p(N_G(P)/P)$ as a virtual $N_G(P)$-module by inflation. Comparing this with the definition of $\mathrm{St}_p(G)$ we see that each of the trivial module and the Steinberg module can be expressed as a linear combination of terms involving only the other. Notice that the only p-subgroups P which make any contribution to the sum in 6.1 are those with $P = O_p(N_G(P))$ because otherwise the Steinberg module of $N_G(P)/P$ is zero, by property 4 in §5. At the end of this section we give an application to cohomology, but first we explain how the formula may be derived.

We work with the precursor of the Steinberg module which lies in the Burnside ring. We define the *Burnside ring*, $b(G) = K_0(G\text{-sets}, 0)$, to be the Grothendieck group of the category of finite G-sets with respect to relations given by disjoint union decompositions. This is the free abelian group with the set of equivalence classes of transitive G-sets as a basis, two transitive G sets $H\backslash G$ and $K\backslash G$ being equivalent if and only if the stabilizer subgroups H and K are conjugate in G. The basis elements thus biject with the conjugacy classes of subgroups of G. If Δ is a simplicial complex of dimension d arising from a poset on which G acts, we let Δ_r denote the G-set of simplices in dimension r, and let $\Delta_{-1} = 1$ be the trivial G-set. We set

$$\Lambda(\Delta) = \sum_{r=0}^{d}(-1)^r\Delta_r,$$

the *Lefschetz invariant* of Δ, as an element of $b(G)$. We also define the *reduced Lefschetz invariant* to be

$$\tilde{\Lambda}(\Delta) = \sum_{r=-1}^{d}(-1)^r\Delta_r.$$

There is a homomorphism $r : b(G) \to a(G)$ sending each G-set to the corresponding permutation module, and evidently if Δ is the simplicial complex of $\mathcal{S}_p(G)$ then $r(\tilde{\Lambda}(\Delta)) = \mathrm{St}_p(G)$. Burnside showed [**5**, §181, p. 238] that a G-set Ω is completely determined as an element of $b(G)$ by knowing the values of $|\Omega^H|$ for all subgroups $H \leq G$. In our situation this means that $\tilde{\Lambda}(\Delta)$ is completely determined by the values of the (reduced) Euler characteristics

$$\tilde{\chi}(\Delta^H) = \sum_{r=-1}^{d}(-1)^r|\Delta_r^H|.$$

Since these numbers are unchanged under G-homotopy equivalence it follows that if Δ_1 and Δ_2 are G-homotopy equivalent then $\tilde{\Lambda}(\Delta_1) = \tilde{\Lambda}(\Delta_2)$. This approach is taken from [**12**]. Thus we have:

6.2. PROPOSITION. $\tilde{\Lambda}(\mathcal{S}_p(G)) = \tilde{\Lambda}(\mathcal{A}_p(G)) = \tilde{\Lambda}(\mathcal{R}_p(G))$. *It makes no difference which of these four simplicial complexes we use to define* $\mathrm{St}_p(G)$.

We now state the inversion formula in the context of the Burnside ring.

6.3. THEOREM (THÉVENAZ [**12**, 3.3]). *Let G act on a poset $\mathcal{P}$. Then*

$$\Lambda(\mathcal{P}) = -\sum_{x\in\mathcal{P}/G} \tilde{\Lambda}(]x,\ [)\uparrow_{G_x}^{G} = -\sum_{x\in\mathcal{P}/G} \tilde{\Lambda}(]\ ,x[)\uparrow_{G_x}^{G}$$

in the Burnside ring.

The notation $]x,\ [$ means the interval $\{y \in \mathcal{P} \mid x < y\}$, and $]\ ,x[$ means $\{y \in \mathcal{P} \mid y < x\}$. In the statement we are not distinguishing between a poset and its associated simplicial complex, because it looks better. Unfortunately, we will need to make this distinction in the proof, and we denote the simplicial complex associated to the poset $\mathcal{P}$ by $\Delta(\mathcal{P})$. If σ is the simplex $x_0 < \cdots < x_n$ we let σ_0 denote the starting vertex x_0.

PROOF.

$$\begin{aligned}\Lambda(\mathcal{P}) &= \sum_{\sigma\in\Delta(\mathcal{P})/G} (-1)^{\dim\sigma} G_\sigma \setminus G \\ &= \sum_{x\in\mathcal{P}/G}\left(\sum_{\{\sigma\in\Delta(\mathcal{P})\mid\sigma_0=x\}/G_x} (-1)^{\dim\sigma} G_\sigma\setminus G_x\right)\uparrow_{G_x}^{G} \\ &= \sum_{x\in\mathcal{P}/G}\left(G_x\setminus G_x - \sum_{\tau\in\Delta(]x,\ [)/Gx} (-1)^{\dim\tau} G_\tau\setminus G_x\right)\uparrow_{G_x}^{G} \\ &= -\sum_{x\in\mathcal{P}/G}\tilde{\Lambda}(]x,\ [)\uparrow_{G_x}^{G}.\end{aligned}$$

For the second equality, apply the first to $\Lambda(\mathcal{P}^{op}) = \Lambda(\mathcal{P})$.

The next lemma is 6.1 of [**7**], except that here we have the extra decoration that the homotopy equivalence is an $N_G(P)$-homotopy equivalence.

6.4. LEMMA. *Whenever P is a p-subgroup of G, $N_G(P)$ acts on the subposet $]P,\ [$ of $\mathcal{S}_p(G)$.*

(a) *There is an $N_G(P)$-homotopy equivalence $]P,\ [\simeq \mathcal{S}_p(N_G(P)/P)$, where we regard the second term as an $N_G(P)$-poset by inflation from $N_G(P)/P$.*

(b) $\tilde{\Lambda}(]P,\ [) = \tilde{\Lambda}(\mathcal{S}_p(N_G(P)/P))$ *in $b(N_G(P))$.*

PROOF. (a) There is an inclusion map

$$i : \mathcal{S}_p(N_G(P)/P) \hookrightarrow]P,\ [$$

given by the bijection between p-subgroups of $N_G(P)/P$ and p-subgroups of $N_G(P)$ containing P. By Bredon's Theorem 2.4 it suffices to show that for each subgroup $H \leq N_G(P)$ the mapping

$$i^H : \mathcal{S}_p(N_G(P)/P)^H \hookrightarrow]P,\ [^H$$

is a homotopy equivalence. We use Quillen's argument in 6.1 of [**7**]. There is a reverse mapping

$$r^H :]P,\ [^H \to \mathcal{S}_p(N_G(P)/P)^H$$

defined as follows: if Q is an H-stable p-subgroup in $]P, [^H$ then $r^H(Q) = (Q \cap N_G(P))/P = N_Q(P)/P$, an H-stable nontrivial p-subgroup. Then $r^H i^H = \mathrm{id}$ and $i^H r^H(Q) \le Q$, so it follows from 1.3 of [7] that r^H and i^H are homotopy equivalences.

(b) is immediate from (a).

6.5. COROLLARY. *In $b(G)$,*

$$G \setminus G = - \sum_{\{1 \le P \le G \mid P \text{a } p\text{-subgroup}\}/G} \tilde{\Lambda}(\mathcal{S}_p(N_G(P)/P)) \uparrow_{N_G(P)}^G .$$

In the above statement, $G \setminus G$ is, of course, the trivial G-set.

PROOF. We deduce 6.5 from 6.4 and 6.3 by taking $\mathcal{P}$ to be the poset of *all* p-subgroups of G, including $\{1\}$. Then $\mathcal{P}$ has a unique minimal element and is contractible, so $\Lambda(\mathcal{P}) = G \setminus G$.

PROOF OF 6.1. Apply the homomorphism $r : b(G) \to a(G)$ to the formula in 6.5.

ALPERIN'S CONJECTURE. We now show how G. Robinson's reformulation of Alperin's conjecture is an instance of the above identities. Let

$l(G) =$ the number of absolutely irreducible p-modular characters of G.

$f_0(G) =$ the number of blocks of defect zero of G.

Alperin has conjectured the statement

$$\text{(A)} \quad l(G) = \sum_{\{1 \le P \le G \mid P \text{a } p\text{-subgroup}\}/G} f_0(N_G(P)/P) \quad \text{for all finite groups } G.$$

Robinson has shown that this is equivalent to the statement

$$\text{(R)} \qquad f_0(G) = l(G) - \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} l(G_\sigma) \quad \text{for all finite groups } G,$$

where Δ is the simplicial complex $\mathcal{R}_p(G)$. Actually, we will see in just a moment that it does not matter if we take Δ to be $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, or $\mathcal{R}_p(G)$. We will deduce the equivalence of (A) and (R) from Corollary 6.5 by regarding l as a function on the Burnside ring. We define

$$l : b(G) \to \mathbf{Z}$$

on the basis elements of $b(G)$ by

$l(H \setminus G) =$ the number of absolutely irreducible p-modular characters of H

and extend by linearity to the whole of $b(G)$. It is immediate that Robinson's reformulation (R) can be stated in the form

$$\text{(R}'\text{)} \qquad f_0(G) = -l(\tilde{\Lambda}(\Delta)) \quad \text{for all finite groups } G,$$

where Δ in the first instance is $\mathcal{R}_p(G)$, but might just as well be $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, or $\mathcal{B}_p(G)$ because these all have the same $\tilde{\Lambda}$ by 6.2. We note that as a function on the Burnside ring, l behaves well with respect to induction and inflation when we are extending by a normal p-subgroup:

6.6. LEMMA. *Let H be a subgroup of G and let P be a normal p-subgroup of G. If $x \in b(H)$ and $y \in b(G/P)$ then $l(x) = l(\mathrm{ind}_H^G x)$ and $l(y) = l(\mathrm{inf}_{G/P}^G y)$.*

PROOF. For induction it is that $\mathrm{ind}_H^G(K \setminus H) = K \setminus G$, and the statement for inflation holds because if $P \leq L \leq G$ then L and L/P have the same number of p-modular irreducibles.

Applying the function l to both sides of the equation in Corollary 6.5, and ignoring inductions and inflations by 6.6, we obtain

6.7. COROLLARY.

$$l(G) = -\sum_{\{1 \leq P \leq G | P \text{a } p\text{-subgroup}\}/G} l(\tilde{\Lambda}(\mathcal{S}_p(N_G(P)/P))).$$

PROOF OF THE EQUIVALENCE OF (A) FROM (R$'$). To deduce (A) from (R$'$) we substitute (R$'$) into the formula of Corollary 6.7.

Now assuming (A) we prove (R$'$) by induction on $|G|$. So assume that (A) holds, and that (R$'$) holds for smaller values of $|G|$. Then Corollary 6.7 gives

$$\begin{aligned} l(G) &= -l(\tilde{\Lambda}(\mathcal{S}_p(G))) - \sum_{\{1 < P \leq G | P \text{a } p\text{-subgroup}\}/G} l(\tilde{\Lambda}(\mathcal{S}_p(N_G(P)/P))) \\ &= -l(\tilde{\Lambda}(\mathcal{S}_p(G))) + \sum_{\{1 < P \leq G | P \text{a } p\text{-subgroup}\}/G} f_0(N_G(P)/P) \quad \text{by induction} \\ &= \sum_{\{1 \leq P \leq G | P \text{a } p\text{-subgroup}\}/G} f_0(N_G(P)/P) \quad \text{assuming (A).} \end{aligned}$$

Since all of the terms on the two sides of the last equality are the same, except for two of them, we deduce

$$f_0(G) = -l(\tilde{\Lambda}(\mathcal{S}_p(G)))$$

as required. This argument also starts the induction when $p \nmid |G|$.

We conclude with an application of Theorem 6.1 to group cohomology.

6.8. THEOREM. *For all $n \in \mathbf{Z}$, for all finitely generated $\mathbf{Z}G$-modules M,*

$$\hat{H}^n(G, M)_p = -\sum_{\{P \leq G | P \text{a } p\text{-subgroup}\}/G} \mathrm{Hom}_{\mathbf{Z}_p N_G(P)}(\mathrm{St}_p(N_G(P)/P), \hat{H}^n(P, M)_p).$$

This equation holds in the Grothendieck group $K_0(\mathrm{Ab}, 0)$ of finitely generated abelian groups with relations given by direct sum decompositions (in the same way as the analogous theorems in [**8**] and [**9**]). Knowledge of all the terms on the right-hand side determines the isomorphism type of the left-hand side. The suffix p is always used to mean completion at p, but in the case of a finite abelian group this is the same as taking the p-torsion subgroup. In the term in parentheses on the right we regard $\hat{H}^n(P, M)$ as an $N_G(P)$-module by the conjugation action. The whole Hom term then counts the multiplicities of the simple modules at the top of $\mathrm{St}_p(G)$ as composition factors in $\hat{H}^n(P, M)_p$, because $\mathrm{St}_p(G)$ is projective, and also for this reason the calculation here may be done with Brauer characters, which suffice to distinguish composition factors. This formula reduces the Tate cohomology of G to the Tate cohomology of the p-subgroups of G (and only the O_p's of parabolic subgroups contribute) together with knowledge of the action of the normalizers of the p-subgroups.

In the statement of 6.8 the functor $\mathrm{Hom}_{\mathbf{Z}_pN_G(P)}(\ ,\hat{H}^n(P,M)_p)$ is applied to a virtual module. In order to make sense of this we have to regard $\mathrm{Hom}(\ ,X)$ as a homomorphism $a(G) \to K_0(\mathrm{Ab},0)$. Thus if U and V are modules then $\mathrm{Hom}(U-V,X)$ is by definition $\mathrm{Hom}(U,X) - \mathrm{Hom}(V,X)$. We follow this convention with other functors throughout the proof of 6.8.

PROOF OF 6.8. For positive values of n we apply the functor $\mathrm{Ext}^n_{\mathbf{Z}_pG}(\ ,M_p)$ to both sides of the equation in Theorem 6.1. For $n \leq 0$ the result follows from the result for positive n by a dimension shifting argument (cf. [**8**, §2]). Suppose $n > 0$. The term on the left of the equation is

$$\mathrm{Ext}^n_{\mathbf{Z}_pG}(\mathbf{Z}_p, M_p) \cong H^n(G,M)_p$$

(see [**8**] for this isomorphism). The typical term on the right is

$$\begin{aligned}
&\mathrm{Ext}^n_{\mathbf{Z}_pG}(\mathrm{St}_p(N_G(P)/P)\uparrow^G_{N_G(P)}, M_p) \cong \mathrm{Ext}^n_{\mathbf{Z}_pN_G(P)}(\mathrm{St}_p(N_G(P)/P), M_p)\\
(*)\qquad &\cong \mathrm{Ext}^n_{\mathbf{Z}_pN_G(P)}(\mathbf{Z}_p, M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*)\\
&\cong H^n(N_G(P), M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*).
\end{aligned}$$

We apply the Lyndon-Hochschild-Serre spectral sequence for the extension $1 \to P \to N_G(P) \to N_G(P)/P \to 1$. The E_2 page is

$$E_2^{r,s} = H^r(N_G(P)/P, H^s(P, M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*)).$$

I claim that as $N_G(P)$-modules with the conjugation (or diagonal) action,

$$H^s(P, M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*) \cong H^s(P,M_p) \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*.$$

This is because if F is one of the modules in a projective resolution of $\mathbf{Z}_p$ over $\mathbf{Z}_pN_G(P)$ then the conjugation action arises from the diagonal action on $\mathrm{Hom}_{\mathbf{Z}_pP}(F, M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*)$, which is isomorphic over $\mathbf{Z}_pN_G(P)$ to $\mathrm{Hom}_{\mathbf{Z}_pP}(F,M_p) \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*$ with the diagonal action because $\mathrm{St}_p(N_G(P)/P)$ is free as a $\mathbf{Z}_p$-module and has the trivial P-action. This isomorphism commutes with taking homology in the definition of $H^*(P,M_p)$ because $\mathrm{St}_p(N_G(P)/P)$ is flat over $\mathbf{Z}_p$. We deduce that $H^s(P, M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*)$ is a (virtual) projective $\mathbf{Z}_pN_G(P)/P$-module. Hence $E_2^{r,s} = 0$ unless $r = 0$ and then

$$\begin{aligned}
E_2^{0,s} &= H^0(N_G(P)/P, H^s(P, M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*))\\
&= \mathrm{Hom}_{\mathbf{Z}_pN_G(P)}(\mathbf{Z}_p, H^s(P,M_p) \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*)\\
&\cong \mathrm{Hom}_{\mathbf{Z}_pN_G(P)}(\mathrm{St}_p(N_G(P)/P), H^s(P,M_p)).
\end{aligned}$$

The spectral sequence stops at the E_2 page, where it is concentrated on the fiber. Hence $E_2^{0,n} \cong H^n(N_G(P), M_p \otimes_{\mathbf{Z}_p} \mathrm{St}_p(N_G(P)/P)^*)$. Putting this together with equation $(*)$ yields the statement of the theorem.

7. Appendix: homotopy equivalent complexes. We present some elementary arguments which show that splitting of a complex and the Lefschetz invariant are independent of G-chain homotopy equivalence. The use of this kind of result is that it helps us to transfer between the complexes $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{B}_p(G)$, and $\mathcal{R}_p(G)$ (or some other such set of complexes that we may happen to have in mind). The independence of the Lefschetz invariant was also shown just before Proposition 6.2, but here we give an alternative algebraic proof. First we clarify the notion of splitting. It is convenient to define it in a general situation where the complex may have nonzero homology. This extends the usual definition for short exact sequences. We work with a chain complex

$$(D., d) = \cdots D_{n+1} \xrightarrow{d_{n+1}} D_n \xrightarrow{d_n} D_{n-1} \cdots$$

of R-modules, where R is some ring. In what follows all homomorphisms are supposed to commute with the action of R.

DEFINITION. The chain complex $(D., d)$ is *split* if there is a map $\alpha : D. \to D.$ of degree $+1$ with $d\alpha d = d$.

7.1. LEMMA. *Let $(D., d)$ be a chain complex of R-modules. The following are equivalent:*

(i) *$(D., d)$ is split.*

(ii) *For every n, in the factorization of d_n as $D_n \xrightarrow{d_n} \operatorname{Im} d_n \overset{i_n}{\hookrightarrow} D_{n-1}$ where i_n is inclusion, both $d_n : D_n \to \operatorname{Im} d_n$ is a split epimorphism and i_n is a split monomorphism.*

(iii) *For every n we can write $D_n = \operatorname{Im} d_{n+1} \oplus \ker d_n / \operatorname{Im} d_{n+1} \oplus \operatorname{Im} d_n$ so that the mapping*

$$\begin{aligned} D_n &= \operatorname{Im} d_{n+1} \oplus \ker d_n / \operatorname{Im} d_{n+1} \oplus \operatorname{Im} d_n \\ &\xrightarrow{d_n} \operatorname{Im} d_n \oplus \ker d_{n-1} / \operatorname{Im} d_n \oplus \operatorname{Im} d_{n-1} = D_{n-1} \end{aligned}$$

is an isomorphism on the summands $\operatorname{Im} d_n$ and is zero on other summands. Thus $D.$ is the splice of split short exact sequences direct sum its homology.

PROOF. (iii)$\Rightarrow$(i). We define α to be the map which is the inverse of d_n on the summands $\operatorname{Im} d_n$ and zero elsewhere.

(ii)$\Rightarrow$(iii). Both $\ker d_n$ and $\operatorname{Im} d_{n+1}$ are summands of D_n. Since $\operatorname{Im} d_{n+1} \subseteq \ker d_n$, this inclusion also splits by the modular law, giving the required decomposition.

(i)$\Rightarrow$(ii). Suppose there exists a map α of degree $+1$ with $d\alpha d = d$. We claim that $\ker(d) = \ker(\alpha d)$ and $\operatorname{Im}(d\alpha) = \operatorname{Im}(d)$ (writing maps on the left). This is because $\ker(d) \subseteq \ker(\alpha d) \subseteq \ker(d\alpha d) = \ker(d)$, and $d(D.) = d\alpha d(D.) \subseteq d\alpha(D.) \subseteq d(D.)$. Now αd and $d\alpha$ are idempotent mappings from $D.$ to itself, since $\alpha(d\alpha d) = \alpha d$ and $(d\alpha d)\alpha = d\alpha$. Therefore the inclusions $\ker(\alpha d) \subseteq D.$ and $d\alpha(D.) \subseteq D.$ are split, and hence so are the inclusions $\ker d \subseteq D.$ and $\operatorname{Im} d \subseteq D.$ The fact that $\ker d \subseteq D.$ splits means also that $D. \to \operatorname{Im} d$ splits, which is what we need.

For any finite-dimensional complex $(D., d)$ of R-modules we define

$$L(D.) = \sum_{i=0}^{n} (-1)^i D_i \in K_0(\operatorname{mod} R, 0)$$

where $n = \dim D.$.

7.2. PROPOSITION. *Let $f : C. \to D.$ be a chain homotopy equivalence of chain complexes of R-modules. Then*
(i) *$C.$ is split if and only if $D.$ is split.*
(ii) $L(C.) = L(D.)$.

PROOF. (i) Suppose that $C.$ is split by a map α of degree $+1$, so $c\alpha c = c$. Let $g : D. \to C.$ be the chain homotopy inverse, and let $t : D. \to D.$ be the R-homomorphism of degree $+1$ which gives $1_{D.} \simeq fg$. The domains and targets of these mappings are as follows:

$$\begin{array}{ccc} C_n & \overset{C_n}{\underset{a_n}{\rightleftarrows}} & C_{n-1} \\ g_n \uparrow\downarrow f_n & & \\ D_n & \overset{d_n}{\underset{t_n}{\rightleftarrows}} & D_{n-1} \end{array}$$

Using the equations

$$c\alpha c = c, \qquad 1 - fg = t\,d + dt, \qquad df = fc, \qquad g\,d = cg, \qquad d^2 = 0,$$

we calculate

$$df\alpha g\,d = fc\alpha cg = fcg = fg\,d = (1 - (t\,d + dt))\,d = d - dt\,d.$$

Thus $\beta = f\alpha g + t$ splits $D.$, since $d\beta\,d = d$.

(ii) (Proof supplied by P. Eccles). We use similar notation to (i), so that $g : D. \to C.$ is the homotopy inverse, $1 - gf = sc + cs$, and $1 - fg = t\,d + dt$. We show that

$$C_0 \oplus D_1 \oplus C_2 \oplus D_3 \oplus \cdots \cong D_0 \oplus C_1 \oplus D_2 \oplus C_3 \oplus \cdots .$$

Let U denote the left-hand side above, and V the right-hand side. We define maps $\theta : U \to V$ and $\phi : V \to U$ by defining them on each summand of U and V. We define

$$\theta(x) = -c_n(x) + f_n(x) + s_{n+1}(x) \in C_{n-1} \oplus D_n \oplus C_{n+1}, \qquad x \in C_n,$$
$$\theta(y) = d_n(y) - g_n(y) - t_{n+1}(y) \in D_{n-1} \oplus C_n \oplus D_{n+1}, \qquad y \in D_n.$$

We define ϕ be the same formulae. Then

$$\begin{aligned}\phi\theta(x_0, y_1, x_2, y_3, \dots) = (&-x_0, -y_1 + \eta_0(x_0), -x_2 + \eta_1'(y_1) + \varsigma_0(x_0), \\ &- y_3 + \eta_2(x_2) + \varsigma_1'(y_1), -x_4 + \eta_3'(y_3) + \varsigma_2(x_2), \dots),\end{aligned}$$

where

$$\eta_n = f_{n+1}s_{n+1} - t_{n+1}f_n, \qquad \varsigma_n = s_{n+2}s_{n+1},$$

and η_n', ς_n' are defined in the same way with the roles of $C.$ and $D.$ reversed. This uses the chain homotopy equivalence, the fact that $c^2 = 0$, and the fact that f and g are chain maps. This means that $\phi\theta$ is an isomorphism. For

$\phi\theta(D_1 \oplus C_2 \oplus \cdots) \subseteq D_1 \oplus C_2 \oplus \cdots$ so that we have a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_1 \oplus C_2 \oplus \cdots & \longrightarrow & C_0 \oplus D_1 \oplus C_2 \oplus \cdots & \longrightarrow & C_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi\theta|} & & \downarrow{\scriptstyle \phi\theta} & & \downarrow{\scriptstyle -1} & & \\
0 & \longrightarrow & D_1 \oplus C_2 \oplus \cdots & \longrightarrow & C_0 \oplus D_1 \oplus C_2 \oplus \cdots & \longrightarrow & C_0 & \longrightarrow & 0.
\end{array}$$

Hence $\phi\theta$ will be an isomorphism provided $\phi\theta|$ is, and an inductive argument gives the result.

Similarly $\theta\phi$ is an isomorphism, and so θ and ϕ are isomorphisms.

REFERENCES

1. S. Bouc, *Modules de Möbius*, C.R. Acad. Sci. Paris Sér. I **299** (1984), 9–12.

2. ——, *Homologie de certains ensembles ordonnés*, C.R. Acad. Sci. Paris Sér. I **299** (1984), 49–52.

3. G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Math., vol. 34, Springer, Berlin, 1967.

4. K. S. Brown, *Cohomology of groups*, Graduate Texts in Math., vol. 87, Springer, New York, 1982.

5. W. Burnside, *Theory of groups of finite order*, 2nd ed., Cambridge Univ. Press, Cambridge, 1911.

6. N. J. Kuhn and S. A. Mitchell, *The multiplicity of the Steinberg representation of* $\mathrm{GL}_n\mathrm{F}_q$ *in the symmetric algebra*, Preprint.

7. D. Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Advances in Math. **28** (1978), 101–128.

8. P. J. Webb, *A local method in group cohomology*, Comment. Math. Helv. (to appear).

9. ——, *Complexes, group cohomology and an induction theorem for the Green ring*, J. Algebra (to appear).

10. ——, *A split exact sequence of cohomology groups* (in preparation).

11. K. S. Brown and J. Thévenaz, *A generalization of Sylow's third theorem* (in preparation).

12. J. Thévenaz, *Permutation representations arising from simplicial complexes*, J. Combin. Theory Ser. A (to appear).

THE UNIVERSITY, MANCHESTER, ENGLAND