Math 8245 **Homework 2, due Monday 13 October** **PJW**

GAP questions:

1. Use GAP to show that

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ca)^5 = 1 \rangle \cong A_5 \times C_2.$$

2. Use GAP to show that $SL(2,5)$ has a normal subgroup of order 2 such that the quotient is isomorphic to A_5. Show that $SL(2,5)$ has no subgroup isomorphic to A_5. Identify the Sylow 2-subgroups of $SL(2,5)$.

3. The generalized quaternion group of order 2^n has a presentation

$$\langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ bab^{-1} = a^{-1} \rangle.$$

   Use GAP to investigate the generalized quaternion group of order 32. Get a list of the orders of the elements. Compute the derived subgroup and the center. Draw a picture of the lattice of subgroups of this group. What is the minimum degree of a faithful permutation representation of this group?

4. Investigate similarly the groups

$$g1 = \langle a, b, c \mid a^3 = b^3 = c^3 = [a, b] = 1,\ cac^{-1} = ab,\ cbc^{-1} = b \rangle,$$
$$g2 = \langle a, b \mid a^9 = b^3 = 1,\ bab^{-1} = a^4 \rangle.$$

   I think the lattices of subgroups are too big to be worth doing, and they do not give much insight. Do you agree?

Theory questions: These are taken from the end of section 2, and have the same numbering as the questions there.

2. Using Exercises 1.5 and 2.1 (which you may assume without proof), show that if k is any field of characteristic 0 then $GL_n(k)$ has only finitely many conjugacy classes of finite subgroups.

3. Let D be a division ring.

(a) Show that the natural $M_n(D)$-module consisting of column vectors of length n is a simple module.

(b) Show that $M_n(D)$ is semisimple and has up to isomorphism only one simple module.

(c) Show that every algebra of the form

$$M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$$

is semisimple.

4. Prove the following extension of 2.4:

THEOREM. *Let A be a finite dimensional semisimple algebra, S a simple A-module and $D = \mathrm{End}_A(S)$. Then S may be regarded as a module over D and the multiplicity of S as a summand of ${}_AA$ equals* $\dim_D S$.

Extra questions: Do **not** hand in.

5. Let s be the Sylow 2-subgroup of the group

$$g = \langle (1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)(4,10,5,6) \rangle.$$

a) Obtain a permutation representation of s on 8 symbols.
b) It is the case that s is isomorphic to one of the groups in question 2. To which one is it isomorphic?
c) Construct a subgroup of g of order divisible by 2, which is not a 2-group and which does not contain a Sylow 2-subgroup of g (any such subgroup will do!).
[This group g is the Mathieu group M_{11}. Use SylowSubgroup, Orbits, Operation.]

6. Repeat question 3. with the group

$$\langle (1,2,3,4,5,6,7,8,9,10,11,12,13), (2,3)(5,10)(7,11)(9,12) \rangle.$$

[This group is $PSL(3,3)$.]

7. Let g be the group generated by the four permutations

$$\begin{aligned}
&(1,2),\\
&(1,3)(2,4),\\
&(1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16), \text{ and}\\
&(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16).
\end{aligned}$$

a) Show that this group is not isomorphic to the Sylow 2-subgroup of A_{16}.
b) How many properties of these two groups can you find which would be the same if the groups were isomorphic, and in this instance are different?
[This problem arose some time ago in discussions between Professors Feshbach and Lannes.]