Math 8245 **Homework 2** **PJW**

Date due: October 2, 2006. Either hand it to me in class or put it in my mailbox by 3:30.

All these questions are to be done using GAP

1. Let $p_1, p_2, p_3, \ldots = 2, 3, 5, \ldots$ be the sequence of primes, and define

$$E_n = p_1 \cdot p_2 \cdots p_n + 1.$$

These are the numbers which appear in Euclid's proof that there are infinitely many primes, and have the property that E_n is not divisible by any of $p_1, \ldots, p_n$. Write a program in GAP which prints out which of the numbers E_n are prime, where $1 \leq n \leq 80$.

[This exercise tests use of loops and the GAP functions for integers. Instead of typing in your program live within a GAP session, you could try creating the program in a separate file in the directory from which you started GAP, and read in the file to GAP using Read("filename"); that way you do not need to type everything again when you make corrections.]

2. Consider the groups:

$$\begin{aligned}
g1 &= \langle (1,2,3,4,5,6,7,8), (2,4)(3,7)(6,8) \rangle \\
g2 &= \langle (1,2,3,4,5,6,7,8), (2,8)(3,7)(4,6) \rangle \\
g3 &= \langle (1,2,3,4,5,6,7,8), (2,6)(4,8) \rangle \\
g4 &= \langle (1,4,6,8,10,12,14,15)(2,3,5,7,9,11,13,16), \\
&\qquad (1,2)(3,15)(4,16)(5,14)(6,13)(7,12)(8,11)(9,10) \rangle \\
g5 &= \langle (1,2,3,4,5,6,7,8), (9,10) \rangle.
\end{aligned}$$

(a) For each group compute

(i) a list of the elements of the group,

(ii) a list of the orders of the elements of the group.

(b) Determine whether any of these groups are isomorphic to one another.

[Computer algorithms which establish an isomorphism between two groups are very poor — they basically run through all possible bijections and see if any of them are group homomorphisms. Instead, to show that certain groups are isomorphic here you should identify the groups in question as groups you already know something about, and use some theory to establish isomorphism.]

(c) In the case of the group $g1$, compute the lattice of subgroups. Show that $g1$ has a subgroup which is cyclic of order 8, another subgroup which is dihedral of order 8, and a third subgroup which is quaternion of order 8. In each case provide generators for the subgroup in question. Draw a picture of the lattice of subgroups, where one

subgroup is shown immediately below another if one is a maximal subgroup of the other — in other words, draw the Hasse diagram.
[There are commands `ConjugacyClassesSubgroups` and `LatticeSubgroups` which I am sure some of you would be tempted to explore in tackling this problem. I suggest that it would be at least as easy for you to construct the lattice of subgroups by intelligent direct use of the functions I have already shown you in GAP. If you do insist on finding out about the subgroup functions I have just mentioned, be warned that I ask for a lattice of subgroups, not of conjugacy classes of subgroups, so I want all subgroups in the picture. Also, I ask for a picture, not the kind of output which these built-in functions produce.]

3. Let s be the Sylow 2-subgroup of the group

$$g = \langle (1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)(4,10,5,6) \rangle.$$

a) Obtain a permutation representation of s on 8 symbols.
b) It is the case that s is isomorphic to one of the groups in question 2. To which one is it isomorphic?
c) Construct a subgroup of g of order divisible by 2, which is not a 2-group and which does not contain a Sylow 2-subgroup of g (any such subgroup will do!).
[This group g is the Mathieu group M_{11}. Use SylowSubgroup, Orbits, Action.]

4. Repeat question 3. with the group

$$\langle (1,2,3,4,5,6,7,8,9,10,11,12,13), (2,3)(5,10)(7,11)(9,12) \rangle.$$

[This group is $PSL(3,3)$.]

5. Use GAP to show that $GL(2,3) \cong Q_8 \rtimes S_3$. Show also that $PSL(2,3) \cong A_4$.
[You may wish to proceed as follows: find the 2-core of $GL(2,3)$ (vocabulary: `PCore`) and show that it is Q_8. Show that the subgroup generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a complement to Q_8 by computing sizes and using `Intersection`. Or work with a permutation representation and use GAP's built-in functions to get a complement.]

6. Using the fact that if $\rho : G \to S_G$ is the embedding of G into permutations of G given by the right regular representation then $N_{S_G}((G)\rho) = (G)\rho \rtimes \mathrm{Aut}(G)$, verify that $\mathrm{Aut}(Q_8) \cong S_4$.
[Vocabulary: `Elements`, `OnRight`, `Action`, `Cosets`. Note that a subgroup of index 4 in S_4 can be obtained as the normalizer of a Sylow 3-subgroup.]

7. Show that the group $\langle (1,5)(2,6), (1,3)(4,6), (2,3)(4,5) \rangle$ is isomorphic to S_4. [This question is needs the same ideas as the final step in question 6, and the second hint to question 6 is relevant.]