Math 8201 **Homework 3** **PJW**

Date due: September 29, 2008 Hand in the 5 starred questions

Section 2.6

M* (a) Show that the group of all motions of the icosahedron is the direct product of a group of order 60 and a group of order 2.
(b) Show that the group of all motions of the tetrahedron cannot be expressed as a direct product of the form $H \times \langle g \rangle$ for any non-identity element g.

N* Let $H \triangleleft G$ and let $\pi : G \to G/H$ be the quotient homomorphism. Suppose that X is a subset of G so that $\pi(X)$ generates G/H. Prove that $G = \langle H \cup X \rangle$.

O Let p be a prime and let H and K be subgroups of a finite G, each of which has order a power of p, and such that H is normal in G.

(a) Show that HK is a subgroup of G whose order is a power of p.

(b) Show that G has a unique largest normal subgroup whose order is a power of p, and that this subgroup contains all other normal subgroups whose order is a power of p. (This subgroup is often denoted $O_p(G)$.) [Use the fact that if in addition that K is normal in G (so now both H and K are normal in G) then HK is normal in G.

(c) Show that the factor group $G/O_p(G)$ has no normal subgroup of order a power of p, apart from the identity subgroup.

Q* (a) Let G be a group of order 24 which has a normal subgroup H of order 8. Show that every element of G not in H has order divisible by 3.
(b) Determine $O_2(S_4)$ (see question O for the definition).

R* Let G be the dihedral group of order 12, which we may regard as the group of isometries of a regular hexagon. Let $\sigma \in G$ be the rotation through an angle of 180° about the midpoint of the hexagon. We have seen that $\langle \sigma \rangle$ is the center of G, and hence is a normal subgroup.
(a) Show that $G/(\sigma) \approx S_3$.
(b) Make a complete list of all subgroups H with $(\sigma) \subseteq H \subseteq G$. For each possible order that H can have, specify how many subgroups there are of that order.

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R Let $a = (1, 2, 3, 4) \in S_4 = G$. Describe the centralizer $C_{S_4}(a)$ and the normalizer $N_{S_4}(\langle a \rangle)$. (Determine their structure and order.)

S* Show that when $a = (4, 5) \in S_5$, the subgroup $C_{S_5}(a)$ consists of $S_3 \cup S_3 a$, where S_3 denotes the symmetric group on three symbols as a subgroup of S_5 permuting the symbols $\{1, 2, 3\}$.

T Compute $C_G(a)$ and $N_G(\langle a \rangle)$ for all elements a in G when G is dihedral of order 10 and of order 12. Compare the answers with the number of conjugates of the element and whether the subgroup is normal.