Math 8202 **Homework 5** **PJW**

Date due: March 2, 2009. Hand in only the 5 starred questions (all of the questions - I apologize for not having a greater selection of questions this week! :-)).

DDD* Suppose $f(x) \in \mathbb{Z}[x]$ is an irreducible quartic whose splitting field has Galois group S_4 over $\mathbb{Q}$ (there are many such quartics). Let θ be a root of $f(x)$ and set $K = \mathbb{Q}(\theta)$. Prove that K is an extension of $\mathbb{Q}$ of degree 4 with no proper subfields. Are there any Galois extensions of $\mathbb{Q}$ of degre 4 with no proper subfields?

EEE* (Fall 2002, qn. 6) Let a be a nonzero rational number.

(a) (6%) Determine the values of a such that the extension $\mathbb{Q}(\sqrt{ai})$ is of degree 4 over $\mathbb{Q}$, where $i^2 = -1$.

(b) (12%) When $K = \mathbb{Q}(\sqrt{ai})$ is of degree 4 over $\mathbb{Q}$ show that K is Galois over $\mathbb{Q}$ with the Klein 4-group as Galois group. In this case determine all the quadratic extensions of $\mathbb{Q}$ contained in K.

FFF* On page 211 of Rotman's book, for a finite extension $K = k(\alpha_1, \ldots, \alpha_n)$ of a field k, a normal closure of K/k is defined to be an extension $E \supseteq K$ of least degree which is the splitting field of some polynomial $f \in k[x]$. No assertion of the uniqueness of E is made. Show that if each of the minimal polynomials of the α_i over k is separable, then the normal closure of K/k is unique.

GGG* Let L be the normal closure of a finite extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$. For any prime p dividing the order of $\mathrm{Gal}(L/\mathbb{Q})$ prove that there is a subfield F of O with $[L : F] = p$ and $L = F(\alpha)$.

HHH* Let F be a subfield of the real numbers $\mathbb{R}$. Let a be an element of F and let $K = F(\sqrt[n]{a})$ where $\sqrt[n]{a}$ denotes a real nth root of a. Prove that if L is any Galois extension of F contained in K then $[L : F] \leq 2$.