Math 8202 **Homework 6** **PJW**

Date due: March 9, 2009. There will be a quiz on this date. Hand in only the 5 starred questions.

Section 4.2, page 247 nos. 4.21*, 4.22

III Determine the Galois groups of the polynomials over $\mathbb{Q}$:

(a) $x^3 - x^2 - 4$

(b) $x^3 - 2x + 4$

(c) $x^3 - x + 1$

(d) $x^3 + x^2 - 2x - 1$.

JJJ* Prove for any $a, b \in \mathbb{F}_{p^n}$ that if $x^3 + ax + b$ is irreducible then $-4a^3 - 27b^2$ is a square in $\mathbb{F}_{p^n}$.

KKK* Let F be an extension of $\mathbb{Q}$ of degree 4 that is not Galois over $\mathbb{Q}$. Prove that the Galois closure of F has Galois group either S_4, A_4 or the dihedral group D_8 of order 8. Prove that the Galois group is dihedral if and only if F contains a quadratic extension of $\mathbb{Q}$.

LLL Prove that an extension F of $\mathbb{Q}$ of degree 4 can be generated by the root of an irreducible biquadratic $x^4 + ax^2 + b$ over $\mathbb{Q}$ if and only if F contains a quadratic extension of $\mathbb{Q}$.

MMM* Let θ be a root of $x^3 - 3x + 1$. Prove that the splitting field of this polynomial is $\mathbb{Q}(\theta)$ and that the Galois group is cyclic of order 3. In particular the other roots of this polynomial can be written in the form $a + b\theta + c\theta^2$ for some $a, , c \in \mathbb{Q}$. Determine the other roots explicitly in terms of θ.

NNN Determine the Galois group of $(x^3 - 2)(x^3 - 3)$ over $\mathbb{Q}$. Determine all the subfields which contain $\mathbb{Q}(\rho)$ where ρ is a primitive 3rd root of unity.

OOO Let α_1, α_2, α_3, α_4 be the roots of a quartic polynomial $f(x)$ over $\mathbb{Q}$. Show that the quantities $\alpha_1\alpha_2 + \alpha_3\alpha_4$, $\alpha_1\alpha_3 + \alpha_2\alpha_4$, $\alpha_1\alpha_4 + \alpha_2\alpha_3$ are the rots of a cubic polynomial with coefficients in $\mathbb{Q}$ (which, like a similar polynomial mentioned in the text, is also sometimes called the resolvent cubic).

PPP (I can't see how to do this one. Perhaps you can?) If $f(x) = x^3 + px + q \in \mathbb{Z}[x]$ is irreducible, prove that its discriminant $D = -4p^3 - 27q^2$ is an integer not equal to $0, \pm 1$.

QQQ* Let F be a field of characteristic 0 in which every cubic polynomial has a root. Let $f(x)$ be an irreducible quartic polynomial over F whose discriminant is a square in F. Determine the Galois group of $f(x)$.

Some topics which we will not do in class appear in exercises from Section 14.6 of the book by Dummit and Foote. Newton's identities appear in questions 22, 23. The fact that every symmetric polynomial can be expressed as a polynomial in the elementary symmetric functions appears in exercises 37 - 43. If you are interested (but it is definitely not on the syllabus) the resultant of two polynomials is discussed following exercise 29, with an application to discriminants in 32. There are more Galois groups of polynomials $x^6 - 2x^3 - 2$ and $x^6 - 4x^3 + 1$ discussed in 48 and 49. In Section 14.2 of Dummit and Foote there is a discussion of properties of the norm and trace following exercise 17.