Math 8202 **Homework 8** **PJW**

Date due: April 6, 2009. There will be a quiz on this date. Hand in only the 5 starred questions.

As announced on the last homework sheet, Quiz 4 will be on April 6, Quiz 5 on April 20 and Quiz 6 on May 4.

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RRR Let M be a left R-module. An element m of M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion submodule* of M).

(b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]

(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

(d) Let $\phi : M \to N$ be an R-module homomorphism. Prove that $\phi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$.

SSS Let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by $\pi/2$ radians. Show that V and 0 are the only $\mathbb{R}[x]$-submodules of V, where the module structure is given by letting x act as T.

TTT Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.

UUU* Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, Z/m\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z}$.

VVV When I is an ideal of R and M is an R-module, let

$$IM = \{\sum_{\text{finite}} a_i m_i \mid a_i \in I,\ m_i \in M\}.$$

For each positive integer n prove that

$$R^n/IR^n \cong R/IR \times \cdots \times R/IR \quad (n \text{ times}).$$

WWW* Let I be a nilpotent ideal in a commutative ring R (in other words, $I^n = 0$ for some n), let M and N be R-modules and let $\phi : M \to N$ be an R-module homomorphism. Show that if the induced map $\bar{\phi} : M/IM \to n/IN$ is surjective, then ϕ is surjective.

XXX Suppose that

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C \\ \alpha\big\downarrow & & \beta\big\downarrow & & \gamma\big\downarrow \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\phi'} & C' \end{array}$$

is a commutative diagram of R-modules and that the rows are exact. Prove that

(a) If ϕ and α are surjective and β is injective then γ is injective

(b) if ψ', α and γ are injective, then β is injective,

(c) if ϕ, α and γ are surjective then β is surjective,

(d) if β is injective, α and γ are surjective then γ is injective,

(e) if β is surjective, γ and ψ' are injective, then α is surjective.

YYY* Let $0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$ be a sequence of R-modules. Prove that the associated sequence

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\phi'} \mathrm{Hom}_R(D, N) \to 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [To show the sequence splits, take $D = N$ and lift the identity map in $\mathrm{Hom}_R(N, N)$.]