Math 8211 **Homework 1** **PJW**

Date due: Monday September 17, 2012. In class on Wednesday September 19 we will grade your answers, so it is important to be present on that day, with your homework.

For extra practice (for example, if you can't see how to get started on the more elaborate questions below) try some further questions from Rotman other than the ones listed below; for example 1.1, 1.5, 1.7.

1. 1.4 of Rotman on page 33.
2. 1.6 of Rotman on page 33.
3. 1.15 of Rotman on page 35.
4. 1.16 of Rotman on page 35.

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism* of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ so that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ so that GF is naturally isomorphic to the identity functor $1_{\mathcal{C}}$ and FG is naturally isomorphic to $1_{\mathcal{D}}$ (meaning that there are invertible natural transformations $\tau : GF \to 1_{\mathcal{C}}$ and $\sigma : FG \to 1_{\mathcal{D}}$).

5. Let G and H be groups, which we may regard as categories with one object in which every morphism is invertible.
   (1) Show that a functor $F : G \to H$ is 'the same as' a group homomorphism $f : G \to H$ when G and H are regarded as groups in the usual way.
   (2) Show that a functor $F : G \to G$ is naturally isomorphic to the identity functor $1_G : G \to G$ if and only if F is an isomorphism of categories of the form $c_g : G \to G$ for some $g \in G$ (regarded as a group homomorphism), where $c_g(h) = ghg^{-1}$.
   (3) Deduce that the group of equivalences $G \to G$ (when G is regarded as a category) is the same as the group of usual isomorphisms $G \to G$.
   (4) Translate the imprecise statement, "a functor $F : G \to H$ is 'the same as' a group homomorphism $f : G \to H$," into a precise statement of category theory.
6. Let I be the poset with two elements 0 and 1, and with $0 < 1$. Recall from 1.12 that if $\mathcal{P}$ and $\mathcal{Q}$ are posets then a functor $\mathcal{P} \to \mathcal{Q}$ is 'the same thing as' an order-preserving map. Consider two order-preserving maps $f, g : \mathcal{P} \to \mathcal{Q}$, and regard them as functors where this makes sense. Show that the following three conditions are equivalent:
   (1) there is a natural transformation $\tau : f \to g$,
   (2) $f(x) \leq g(x)$ for all $x \in \mathcal{P}$,
   (3) there is a functor $h : \mathcal{P} \times I \to \mathcal{Q}$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \mathcal{P}$.
7. Given a poset $\mathcal{P}$ we may construct an abstract simplicial complex $|\mathcal{P}|$ (see page 11) where the n-simplices are the sets of elements $\{x_0, \ldots, x_n\} \subseteq \mathcal{P}$ such that (possibly after reordering) $x_0 < x_1 < \cdots < x_n$.

(1) Applying this to the poset $I \times I$, how many simplices does the simplicial complex $|I \times I|$ have? When $|I \times I|$ is realized as a topological space, what is its dimension? Draw a picture of this space.

(2) Show that the construction $|-|$ allows us to define a functor from the category of posets with order-preserving maps as morphisms, to the category of abstract simplicial complexes with simplicial maps as morphisms.

(3) Suppose that there is a natural transformation $\tau : f \to g$ where $f, g : \mathcal{P} \to \mathcal{Q}$ are order-preserving maps between posets. Show that the functorially induced maps $|f|, |g| : |\mathcal{P}| \to |\mathcal{Q}|$ are homotopic. (For this you may assume that $|\mathcal{P} \times I|$ is homeomorphic to $|\mathcal{P}| \times |I|$ when these are regarded as topological spaces.)

(4) Suppose that $\mathcal{P}$ is a poset with a unique maximal element. Show that $|\mathcal{P}|$ is contractible (that is, homotopy equivalent to a point).

to each continuous map $X \to Y$, assign the chain map $f_\#\colon \mathbf{S}_\bullet(X) \to \mathbf{S}_\bullet(Y)$. The nth homology functor is the composite **Top** $\to$ **Comp** $\to$ **Ab**, where **Comp** $\to$ **Ab** is defined by $\mathbf{S}_\bullet(X) \mapsto H_n(X)$ and $f_\# \mapsto H_n(f)$. Thus, in a very precise way, we see that homology has a topological half and an algebraic half, for the functor **Top** $\to$ **Comp** involves the topological notions of spaces and continuous maps, while the functor **Comp** $\to$ **Ab** involves only algebra. Homological Algebra is the study of this algebraic half.

Exercises

***1.1** **(i)** Prove, in every category $\mathcal{C}$, that each object $A \in \mathcal{C}$ has a unique identity morphism.

(ii) If f is an isomorphism in a category, prove that its inverse is unique.

1.2 **(i)** Prove that there is a functor $F\colon$ **ComRings** $\to$ **ComRings** defined on objects by $F\colon R \mapsto R[x]$ and on morphisms $\varphi\colon R \to S$ by $F\varphi\colon r_0 + r_1x + \cdots + r_nx^n \mapsto \varphi(r_0) + \varphi(r_1)x + \cdots + \varphi(r_n)x^n$.

(ii) Prove that there is a functor on **Dom**, the category of all (integral) domains, defined on objects by $R \mapsto \operatorname{Frac}(R)$, and on morphisms $f\colon R \to S$ by $r/1 \mapsto f(r)/1$.

1.3 Let $\mathcal{A} \xrightarrow{S} \mathcal{B} \xrightarrow{T} \mathcal{C}$ be functors. If the variances of S and T are the same, prove that the composite $TS\colon \mathcal{A} \to \mathcal{C}$ is a covariant functor; if the variances of S and T are different, prove that TS is a contravariant functor

***1.4** If $T\colon \mathcal{A} \to \mathcal{B}$ is a functor, define $T^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ by $T^{\mathrm{op}}(A) = T(A)$ for all $A \in \operatorname{obj}(\mathcal{A})$ and $T^{\mathrm{op}}(f^{\mathrm{op}}) = T(f)$ for all morphisms f in $\mathcal{A}$. Prove that T^{op} is a functor having variance opposite to the variance of T.

1.5 **(i)** If X is a set and k is a field, define the vector space k^X to be the set of all functions $X \to k$ under pointwise operations. Prove that there is a functor $G\colon$ **Sets** $\to {}_k$**Mod** with $G(X) = k^X$.

(ii) Define $U\colon {}_k$**Mod** $\to$ **Sets** to be the forgetful functor [see Example 1.9(v)]. What are the composites $GU\colon {}_k$**Mod** $\to {}_k$**Mod** and $UG\colon$ **Sets** $\to$ **Sets**?

***1.6** **(i)** If X is a set, define FX to be the free group having basis X; that is, the elements of FX are reduced words on the alphabet X and multiplication is juxtaposition followed by cancellation. If $\varphi\colon X \to Y$ is a function, prove that there is a unique homomorphism $F\varphi \colon FX \to FY$ such that $(F\varphi)|X = \varphi$.

(ii) Prove that the $F\colon$ **Sets** $\to$ **Groups** is a functor (F is called the ***free functor***).

***1.7** **(i)** Define $\mathcal{C}$ to have objects all ordered pairs (G, H), where G is a group and H is a normal subgroup of G, and to have morphisms $\varphi_* \colon (G, H) \to (G_1, H_1)$, where $\varphi \colon G \to G_1$ is a homomorphism with $\varphi(H) \subseteq H_1$. Prove that $\mathcal{C}$ is a category if composition in $\mathcal{C}$ is defined to be ordinary composition.

(ii) Construct a functor $Q \colon \mathcal{C} \to$ **Groups** with $Q(G, H) = G/H$.

(iii) Prove that there is a functor **Groups** $\to$ **Ab** taking each group G to G/G', where G' is its commutator subgroup.

1.8 If X is a topological space, define $C(X)$ to be its ring of continuous real-valued functions, $C(X) = \{f \colon X \to \mathbb{R} : f \text{ is continuous}\}$, under pointwise operations: $f + g \colon x \mapsto f(x) + g(x)$ and $fg \colon x \mapsto f(x)g(x)$. Prove that there is a contravariant functor $T\colon$ **Top** $\to$ **ComRings** with $T(X) = C(X)$. [A theorem of Gelfand and Kolmogoroff (see Dugundji, *Topology*, p. 289) says that if X and Y are compact Hausdorff spaces and the rings $C(X)$ and $C(Y)$ are isomorphic, then the spaces X and Y are homeomorphic.]

1.9 Let X be a set and let $\mathcal{B}(X)$ be the ***Boolean ring*** whose elements are the subsets of X, whose multiplication is intersection, and whose addition is symmetric difference: if $A, B \subseteq X$, then $AB = A \cap B$, $A + B = (A - B) \cup (B - A)$, and $-A = A$. You may assume that $\mathcal{B}(X)$ is a commutative ring under these operations in which $\varnothing$ is the zero element and X is the 1 element.

(i) Prove that $\mathcal{B}(X)$ has only one unit (recall that an element $u \in R$ is a ***unit*** if there is $v \in R$ with $uv = 1 = vu$).

(ii) If $Y \subsetneq X$ is a proper subset of X, prove that $\mathcal{B}(Y)$ is *not* a subring of $\mathcal{B}(X)$.

(iii) Prove that a nonempty subset $I \subseteq \mathcal{B}(X)$ is an ideal if and only if $A \in I$ implies that every subset of A also lies in I. In particular, the principal ideal (A) generated by a subset A is the family of all the subsets of A.

(iv) Prove that every maximal ideal M in $\mathcal{B}(X)$ is a principal ideal of the form $(X - \{x\})$ for some $x \in X$.

(v) Prove that every prime ideal in $\mathcal{B}(X)$ is a maximal ideal.

1.10** Let R be a ring. Call an (additive) abelian group M an ***almost left R-module if there is a function $R \times M \to M$ satisfying all the axioms of a left R-module except axiom (iv): we do not assume that $1m = m$ for all $m \in M$. Prove that $M = M_1 \oplus M_0$ (direct sum of abelian groups), where $M_1 = \{m \in M : 1m = m\}$ and $M_0 = \{m \in M : rm = 0 \text{ for all } r \in R\}$ are subgroups of M that are almost left R-modules; in fact, M_1 is a left R-module.

***1.11** Prove that every right R-module is a left R^{op}-module, and that every left R-module is a right R^{op}-module.

1.12 If R and A are rings, an ***anti-homomorphism*** $\varphi : R \to A$ is an additive function for which $\varphi(rr') = \varphi(r')\varphi(r)$ for all $r, r' \in R$.

- **(i)** Prove that R and A are anti-isomorphic if and only if $A \cong R^{\text{op}}$.
- **(ii)** Prove that transposition $B \mapsto B^T$ is an anti-isomorphism of a matrix ring $\text{Mat}_n(R)$ with itself, where R is a commutative ring. (If R is not commutative, then $B \mapsto B^T$ is an isomorphism $[\text{Mat}_n(R)]^{\text{op}} \cong \text{Mat}_n(R^{\text{op}})$.)

1.13** An R-map $f : M \to M$, where M is a left R-module, is called an ***endomomorphism.

- **(i)** Prove that $\text{End}_R(M) = \{f : M \to M : f \text{ is an } R\text{-map}\}$ is a ring (under pointwise addition and composition as multiplication) and that M is a left $\text{End}_R(M)$-module. We call $\text{End}_R(M)$ the ***endomorphism ring*** of M.
- **(ii)** If a ring R is regarded as a left R-module, prove that there is an isomorphism $\text{End}_R(R) \to R^{\text{op}}$ of rings.

1.14

- **(i)** Give an example of topological spaces X, Y and an injective continuous map $i : X \to Y$ whose induced map $H_n(i) : H_n(X) \to H_n(Y)$ is not injective for some $n \geq 0$.

  Hint. You may assume that $H_1(S^1) \cong \mathbb{Z}$.
- **(ii)** Give an example of a subspace $A \subseteq X$ of a topological space X and a continuous map $f : X \to Y$ such that $H_n(f) \neq 0$ for some $n \geq 0$ and $H_n(f|A) = 0$.

***1.15** Let $F, G : \mathcal{A} \to \mathcal{B}$ and $F', G' : \mathcal{B} \to \mathcal{C}$ be functors of the same variance, and let $\tau : F \to G$ and $\tau' : F' \to G'$ be natural transformations.

- **(i)** Prove that their composite $\tau'\tau$ is a natural transformation $F'F \to G'G$ where, for each $A \in \text{obj}(\mathcal{A})$, we define

$$(\tau'\tau)_A = \tau'_{FA}\tau_A : F'F(A) \to G'G(A).$$

- **(ii)** If $\tau : F \to G$ is a natural isomorphism, define $\sigma_C : FC \to GC$ for all $C \in \text{obj}(\mathcal{A})$ by $\sigma_C = \tau_C^{-1}$. Prove that σ is a natural transformation $G \to F$.

***1.16** Let $\mathcal{C}$ be a category and let $A, B \in \text{obj}(\mathcal{C})$. Prove the converse of Corollary 1.18: if $A \cong B$, then $\text{Hom}_{\mathcal{C}}(A, \square)$ and $\text{Hom}_{\mathcal{C}}(B, \square)$ are naturally isomorphic functors.
Hint. If $\alpha \colon A \to B$ is an isomorphism, define $\tau_C = (\alpha^{-1})^*$.

1.17 **(i)** Let $\mathcal{A}$ be the category with $\text{obj}(\mathcal{A}) = \{A, B, C, D\}$ and morphisms $\text{Hom}_{\mathcal{A}}(A, B) = \{f\}$, $\text{Hom}_{\mathcal{A}}(C, D) = \{g\}$, and four identities. Define $F \colon \mathcal{A} \to$ **Sets** by $F(A) = \{1\}$, $F(B) = \{2\} = F(C)$, and $F(D) = \{3\}$. Prove that F is a functor but that $\text{im}\, F$ is not a subcategory of **Sets**.

Hint. The composite $Fg \circ Ff$, which is defined in **Sets**, does not lie in $\text{im}\, F$.

(ii) Prove that if $F \colon \mathcal{A} \to \mathcal{B}$ is a functor with $F|\text{obj}(\mathcal{A})$ an injection, then $\text{im}\, F$ is a subcategory of $\mathcal{B}$.

1.18 Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Prove that $\mathcal{A} \times \mathcal{B}$ is a category, where $\text{obj}(\mathcal{A} \times \mathcal{B}) = \text{obj}(\mathcal{A}) \times \text{obj}(\mathcal{B})$, where $\text{Hom}_{\mathcal{A}\times\mathcal{B}}\big((A_1, B_1), (A_2, B_2)\big)$ consists of all $(f, g) \in \text{Hom}_{\mathcal{A}}(A_1, A_2) \times \text{Hom}_{\mathcal{B}}(B_1, B_2)$, and where composition is $(f', g')(f, g) = (f'f, g'g)$.

***1.19** **(i)** If $\mathcal{A}$ is a small category and $F, G \colon \mathcal{A} \to \mathcal{B}$ are functors of the same variance, prove that $\text{Nat}(F, G)$ is a set (not a proper class).

(ii) Give an example of categories $\mathcal{C}, \mathcal{D}$ and functors $S, T : \mathcal{C} \to \mathcal{D}$ such that $\text{Nat}(S, T)$ is a proper class. [As discussed on page 8, do not worry whether $\text{Nat}(S, T)$ is a class.]

Hint. Let $\mathcal{S}$ be a discrete subcategory of **Sets**, and consider $\text{Nat}(T, T)$, where $T : \mathcal{S} \to$ **Sets** is the inclusion functor.

1.20 Show that **Cat** is a category, where obj(**Cat**) is the class of all *small* categories, where $\text{Hom}_{\mathbf{Cat}}(\mathcal{A}, \mathcal{B}) = \mathcal{B}^{\mathcal{A}}$, and where composition is the usual composition of functors. [We assume that categories here are small in order that $\text{Hom}_{\mathbf{Cat}}(\mathcal{A}, \mathcal{B})$ be a set.]