Math 8211 **Homework 4** **PJW**

Date due: Monday November 12, 2012. In class on Wednesday November 14 we will grade your answers, so it is important to be present on that day, with your homework.

Rotman 6.6, 6.8 (page 339), 6.12 (page 340) 6.13 (page 376) .
Questions 1 – 3 below.

1. Let $R = k[X]/(X^3)$ where k is a field. Let C be the complex $R \xrightarrow{X^2} R$.
   (a) Find $\dim_k \operatorname{Hom}_{R\,\text{Comp}}(C, C)$, the dimension of the space of chain maps from C to C.
   (b) Find the dimension of the subspace of chain maps $C \to C$ which are homotopic to zero. Hence find the dimension of the space $\operatorname{Hom}_{R\,\text{HoComp}}(C, C)$ of homotopy classes of chain maps $C \to C$.
   (c) Show that, for this complex C, the set of chain maps $C \to C$ which are non-isomorphisms forms a vector subspace of the space of all endomorphisms of C. Find the dimension of this subspace.
   (d) Show that it is possible to find another complex D for which the set of non-isomorphisms $D \to D$ does not form a vector subspace of all endomorphisms.
   (e) Show that, for this complex C, all chain maps $C \to C$ which are equivalences are, in fact, automorphisms
   (f) Determine, for this complex C, whether or not all invertible chain maps $C \to C$ are homotopic to each other.

2. (a) Suppose that U, V, and W are R-modules and that there are homomorphisms

$$U \underset{\delta}{\overset{\alpha}{\rightleftarrows}} V \underset{\gamma}{\overset{\beta}{\rightleftarrows}} W$$

   such that $\beta\alpha = 0$ and such that the identity map on V can be written $1_V = \alpha\delta + \gamma\beta$. Show that $\beta = \beta\gamma\beta$. Suppose in addition to all this that $\alpha = \alpha\delta\alpha$. Show that $V \cong \alpha\delta(V) \oplus \gamma\beta(V)$.
   (b) Recall that a chain complex C of R-modules is called *contractible* if it is chain homotopy equivalent to the zero chain complex. Prove that C is contractible if and only if C can be written as a direct sum of chain complexes of the form $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$ where α is an isomorphism.

3. (a) Suppose that we have chain maps $C \xrightarrow{f} D \xrightarrow{g} E$ and suppose that D is a contractible complex. Show that the composite gf is homotopic to zero (i.e. null homotopic).
(b) By considering the diagram

$$\begin{array}{ccccccccccc}
C: & \cdots & \xrightarrow{d} & C_2 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_0 & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle i_C} & & & \downarrow{\scriptstyle \binom{d}{1}} & & \downarrow{\scriptstyle \binom{d}{1}} & & \downarrow{\scriptstyle \binom{d}{1}} & & \\
I_C: & \cdots & \xrightarrow{\delta} & C_1 \oplus C_2 & \xrightarrow{\delta} & C_0 \oplus C_1 & \xrightarrow{\delta} & C_1 \oplus C_0 & \xrightarrow{\delta} & \cdots
\end{array}$$

where $\delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ show that any complex complex C can be embedded in a contractible complex I_C.
(c) Show that if $f = Td + eT : C \to D$ is any null-homotopic map of complexes then f defines a chain map $I_C \to D$ as follows:

$$\begin{array}{ccccccccccc}
I_C: & \cdots & \xrightarrow{\delta} & C_1 \oplus C_2 & \xrightarrow{\delta} & C_0 \oplus C_1 & \xrightarrow{\delta} & C_1 \oplus C_0 & \xrightarrow{\delta} & \cdots \\
\downarrow & & & \downarrow{\scriptstyle (T,eT)} & & \downarrow{\scriptstyle (T,eT)} & & \downarrow{\scriptstyle (T,eT)} & & \\
D: & \cdots & \xrightarrow{e} & D_2 & \xrightarrow{e} & D_1 & \xrightarrow{e} & D_0 & \xrightarrow{e} & \cdots
\end{array}$$

such that the composite of this morphism with i_C is f. Deduce that any null-homotopic map factors through a contractible complex.

Extra questions: do not hand in!

Given a homomorphism of chain complexes of R-modules $\phi : \mathcal{C} \to \mathcal{D}$ we may define $E_n = C_{n-1} \oplus D_n$, and a mapping $e_n : E_n \to E_{n-1}$ by $e_n(a,b) = (-\partial a, \phi a + \partial b)$, where we denote the boundary maps on $\mathcal{C}$ and $\mathcal{D}$ by ∂. The specification $\mathcal{E}(\phi) = \{E_n, e_n\}$ is called the *mapping cone* of ϕ.

A. Show that $\mathcal{E} = \{E_n, e_n\}$ is indeed a chain complex.

B. Show that there is a short exact sequence of chain complexes $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{C}[1] \to 0$ where $\mathcal{C}[1]$ denotes the chain complex with the same R-modules and boundary maps as $\mathcal{C}$ but with the labeling of degrees shifted by 1 in an appropriate direction. Deduce that there is a long exact sequence

$$\cdots \to H_n(\mathcal{C}) \to H_n(\mathcal{D}) \to H_n(\mathcal{E}(\phi)) \to H_{n-1}(\mathcal{C}) \to \cdots$$

Show that $\mathcal{E}(\phi)$ is acyclic if and only if ϕ induces an isomorphism $H_n(\mathcal{C}) \to H_n(\mathcal{D})$ for every n.

C. Show that if $\phi \simeq \psi : \mathcal{C} \to \mathcal{D}$ then $\mathcal{E}(\phi) \cong \mathcal{E}(\psi)$.

prove that the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccc} & A'_n/\operatorname{im} d'_{n+1} & \to & A_n/\operatorname{im} d_{n+1} & \to & A''_n/\operatorname{im} d''_{n+1} & \to 0 \\ & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & \\ 0 \longrightarrow & \ker d'_{n-1} & \longrightarrow & \ker d_{n-1} & \longrightarrow & \ker d''_{n-1}. & \end{array}$$

(iv) Use part (ii) and this last diagram to give another proof of Theorem 6.10, the Long Exact Sequence.

6.6 Let $f, g\colon \mathbf{C} \to \mathbf{C}'$ be chain maps, and let $F\colon \mathcal{C} \to \mathcal{C}'$ be an additive functor. If $f \simeq g$, prove that $Ff \simeq Fg$; that is, if f and g are homotopic, then Ff and Fg are homotopic.

***6.7** Let $0 \to \mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'' \to 0$ be an exact sequence of complexes in which $\mathbf{C}'$ and $\mathbf{C}''$ are acyclic; prove that $\mathbf{C}$ is also acyclic.

6.8 Let R and A be rings, and let $T\colon {}_R\mathbf{Mod} \to {}_A\mathbf{Mod}$ be an exact additive functor. Prove that T commutes with homology; that is, for every complex $(\mathbf{C}, d) \in {}_R\mathbf{Comp}$ and for every $n \in \mathbb{Z}$, there is an isomorphism

$$H_n(T\mathbf{C}, Td) \cong TH_n(\mathbf{C}, d).$$

***6.9** **(i)** Prove that homology commutes with direct sums: for all n, there are natural isomorphisms

$$H_n\Big(\bigoplus_\alpha \mathbf{C}^\alpha\Big) \cong \bigoplus_\alpha H_n(\mathbf{C}^\alpha).$$

(ii) Define a direct system of complexes $(\mathbf{C}^i)_{i\in I}$, $(\varphi^i_j)_{i\le j}$, and prove that $\varinjlim \mathbf{C}^i$ exists.

(iii) If $(\mathbf{C}^i)_{i\in I}$, $(\varphi^i_j)_{i\le j}$ is a direct system of complexes over a directed index set, prove, for all $n \ge 0$, that

$$H_n(\varinjlim \mathbf{C}^i) \cong \varinjlim H_n(\mathbf{C}^i).$$

***6.10** Assume that a complex $(\mathbf{C}, d)$ of R-modules has a contracting homotopy in which the maps $s_n\colon C_n \to C_{n+1}$ satisfying

$$1_{C_n} = d_{n+1}s_n + s_{n-1}d_n$$

are only $\mathbb{Z}$-maps. Prove that $(\mathbf{C}, d)$ is an exact sequence.

6.11** (Barratt–Whitehead***). Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc} \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \xrightarrow{\partial_n} & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow \\ & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle g_{n-1}} & & \downarrow{\scriptstyle h_{n-1}} & \\ \longrightarrow & A'_n & \xrightarrow[j_n]{} & B'_n & \xrightarrow[q_n]{} & C'_n & \longrightarrow & A'_{n-1} & \longrightarrow & B'_{n-1} & \longrightarrow & C'_{n-1} & \longrightarrow . \end{array}$$

If each h_n is an isomorphism, prove that there is an exact sequence

$$\to A_n \xrightarrow{(f_n, i_n)} A'_n \oplus B_n \xrightarrow{j_n - g_n} B'_n \xrightarrow{\partial_n h_n^{-1} q_n} A_{n-1} \\ \to A'_{n-1} \oplus B_{n-1} \to B'_{n-1} \to,$$

where

$$(f_n, i_n) \colon a_n \mapsto (f_n a_n, i_n a_n) \text{ and } j_n - g_n \colon (a'_n, b_n) \mapsto j_n a'_n - g_n b_n.$$

6.12** (Mayer–Vietoris***). Given a commutative diagram of complexes with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C}' & \xrightarrow{i} & \mathbf{C} & \xrightarrow{p} & \mathbf{C}'' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & \mathbf{A}' & \xrightarrow[j]{} & \mathbf{A} & \xrightarrow[q]{} & \mathbf{A}'' & \longrightarrow & 0, \end{array}$$

if every third vertical map h_* in the diagram

$$\begin{array}{ccccccccc} \longrightarrow H_n(\mathbf{C}') & \xrightarrow{i_*} & H_n(\mathbf{C}) & \xrightarrow{p_*} & H_n(\mathbf{C}'') & \xrightarrow{\partial} & H_{n-1}(\mathbf{C}') \longrightarrow \\ \downarrow f_* & & \downarrow g_* & & \downarrow h_* & & \downarrow f_* \\ \longrightarrow H_n(\mathbf{A}') & \xrightarrow[j_*]{} & H_n(\mathbf{A}) & \xrightarrow[q_*]{} & H_n(\mathbf{A}'') & \xrightarrow[\partial']{} & H_{n-1}(\mathbf{A}') \longrightarrow \end{array}$$

is an isomorphism, prove that there is an exact sequence

$$\to H_n(\mathbf{C}') \to H_n(\mathbf{A}') \oplus H_n(\mathbf{C}) \to H_n(\mathbf{A}) \to H_{n-1}(\mathbf{C}') \to \cdot$$

6.2 Derived Functors

In order to apply the general results in the previous section, we need a source of short exact sequences of complexes. The idea is to replace every module by a deleted resolution of it; given a short exact sequence of modules, we shall see that this replacement gives a short exact sequence of complexes. We then apply either Hom or $\otimes$, and the resulting homology modules are called Ext or Tor.

We know that a module has many presentations; since resolutions are generalized presentations, the next result is fundamental.

Theorem 6.16 (Comparison Theorem). *Let $\mathcal{A}$ be an abelian category. Given a morphism $f \colon A \to A'$ in $\mathcal{A}$, consider the diagram*

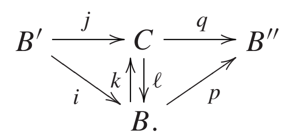
$$\begin{array}{ccccccccc} \longrightarrow P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ \downarrow \check{f}_2 & & \downarrow \check{f}_1 & & \downarrow \check{f}_0 & & \downarrow f & & \\ \longrightarrow P'_2 & \xrightarrow[d'_2]{} & P'_1 & \xrightarrow[d'_1]{} & P'_0 & \xrightarrow[\varepsilon']{} & A' & \longrightarrow & 0, \end{array}$$

where $\partial_n = \sum_{i=0}^{n}(-1)^i d_n^i$. The given identities for d_n^i imply $\partial\partial = 0$. Thus, simplicial objects have homology. The degeneracies allow one to construct an abstract version of homotopy groups as well (see Gelfand–Manin, *Methods of Homological Algebra*, May, *Simplicial Objects in Algebraic Topology*, and Weibel, *An Introduction to Homological Algebra*).

Exercises

6.13 If $\tau\colon F \to G$ is a natural transformation between additive functors, prove that τ gives chain maps $\tau_{\mathbf{C}}\colon F\mathbf{C} \to G\mathbf{C}$ for every complex $\mathbf{C}$. If τ is a natural isomorphism, prove that $F\mathbf{C} \cong G\mathbf{C}$.

***6.14** Consider the commutative diagram with exact row



If k is an isomorphism with inverse ℓ, prove exactness of

$$B' \xrightarrow{i} B \xrightarrow{p} B''.$$

6.15 Let $T\colon \mathcal{A} \to \mathcal{C}$ be an exact additive functor between abelian categories, and suppose that P projective implies TP projective. If $B \in \text{obj}(\mathcal{A})$ and $\mathbf{P}_B$ is a deleted projective resolution of B, prove that $T\mathbf{P}_{TB}$ is a deleted projective resolution of TB.

6.16 Let R be a k-algebra, where k is a commutative ring, which is flat as a k-module. Prove that if B is an R-module (and hence a k-module), then

$$R \otimes_k \text{Tor}_n^k(B, C) \cong \text{Tor}_n^R(B, R \otimes_k C)$$

for all k-modules C and all $n \geq 0$.

6.17 Let R be a semisimple ring.

(i) Prove, for all $n \geq 1$, that $\text{Tor}_n^R(A, B) = \{0\}$ for all right R-modules A and all left R-modules B.

(ii) Prove, for all $n \geq 1$, that $\text{Ext}_R^n(A, B) = \{0\}$ for all left R-modules A and B.

***6.18** If R is a PID, prove, for all $n \geq 2$, that $\text{Tor}_n^R(A, B) = \{0\} = \text{Ext}_R^n(A, B)$ for all R-modules A and B.
Hint. Use Corollary 4.15.

***6.19** Let R be a domain with fraction field Q, and let A, C be R-modules. If either C or A is a vector space over Q, prove that $\text{Tor}_n^R(C, A)$ and $\text{Ext}_R^n(C, A)$ are also vector spaces over Q.
Hint. Use Exercise 2.38 on page 97.