Date due: Wednesday February 20, 2013. In class on February 22 we will grade your answers, so it is important to be present on that day, with your homework.

Exercises to §3. Prove the following propositions.

3.1. Let $I_1, \ldots, I_n$ be ideals of a ring A such that $I_1 \cap \cdots \cap I_n = (0)$; if each A/I_i is a Noetherian ring then so is A.

3.2. Let A and B be Noetherian rings, and $f: A \longrightarrow C$ and $g: B \longrightarrow C$ ring homomorphisms. If both f and g are surjective then the fibre product $A \times_C B$ (that is, the subring of the direct product $A \times B$ given by $\{(a,b) \in A \times B \mid f(a) = g(b)\}$ is a Noetherian ring.

3.3. Let A be a local ring such that the maximal ideal $\mathfrak{m}$ is principal and $\bigcap_{n>0} \mathfrak{m}^n = (0)$. Then A is Noetherian, and every non-zero ideal of A is a power of $\mathfrak{m}$.

3.4. Let A be an integral domain with field of fractions K. A *fractional ideal* I of A is an A-submodule I of K such that $I \neq 0$ and $\alpha I \subset A$ for some $0 \neq \alpha \in K$. The product of two fractional ideals is defined in the same way as the product of two ideals. If I is a fractional ideal of A we set $I^{-1} = \{\alpha \in K \mid \alpha I \subset A\}$; this is also a fractional ideal, and $II^{-1} \subset A$. In the particular case that $II^{-1} = A$ we say that I is *invertible*. An invertible fractional ideal of A is finitely generated as an A-module.

3.5. If A is a UFD, the only ideals of A which are invertible as fractional ideals are the principal ideals.

3.6. Let A be a Noetherian ring, and $\varphi: A \longrightarrow A$ a homomorphism of rings. Then if φ is surjective it is also injective, and hence an automorphism of A.

3.7. If A is a Noetherian ring then any finite A-module is of finite presentation, but if A is non-Noetherian then A must have finite A-modules which are not of finite presentation.