Math 8212 **Homework 3** **PJW**

Date due: Wednesday March 6, 2013. In class on Friday March 8 we will grade your answers, so it is important to be present on that day, with your homework.

Questions 4.1, 4.2, 4.3, 4.5, 4.8, 4.9, 4.10 on page 29 of Matsumura.

Extra questions:

The questions I missed out from section 4 could be taken as extra questions, and some of them are fairly easy, so could be used as warm-up questions for those people who find that helpful. Questions 4.4 and 4.7 are easy, I think. If you try 4.6 it would be helpful to do 2.1 first. In any case, 2.1 is a good question

Here are a couple more.

1. Let $\mathfrak{m}$ be a maximal ideal and suppose that the ideal I contains $\mathfrak{m}^\nu$ for some ν. Show that $\mathfrak{m}$ is the only maximal ideal which contains I.
2. Give an example of a ring with ideals I_j for which $V(\bigcap I_j) \neq \bigcup V(I_j)$.

(i) with $r=0$, we have that $K_{\mathfrak{q}}=0$ for every $\mathfrak{q}$ in a neighbourhood V of $\mathfrak{p}$; this gives $(A_{\mathfrak{q}})^r \simeq M_{\mathfrak{q}}$, so that $V \subset U_F$. ■

Exercise to §4. Prove the following propositions.

4.1. The radical of a primary ideal is prime; also, if I is a proper ideal containing a power $\mathfrak{m}^\nu$ of a maximal ideal $\mathfrak{m}$ then I is primary and $\sqrt{I}=\mathfrak{m}$.

4.2. If P is a prime ideal of a ring A then the *symbolic nth power* of P is the ideal $P^{(n)}$ given by

$P^{(n)} = P^n A_P \cap A.$

This is a primary ideal with radical P.

4.3. If S is a multiplicative set of a ring A then $\operatorname{Spec}(A_S)$ is homeomorphic to the subspace $\{\mathfrak{p} \mid \mathfrak{p} \cap S = \varnothing\} \subset \operatorname{Spec} A$; this is in general neither open nor closed in $\operatorname{Spec} A$.

4.4. If I is an ideal of A then $\operatorname{Spec}(A/I)$ is homeomorphic to the closed subset $V(I)$ of $\operatorname{Spec} A$.

4.5. The spectrum of a ring $\operatorname{Spec} A$ is quasi-compact, that is, given an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of $X=\operatorname{Spec} A$ (with $X=\bigcup_\lambda U_\lambda$), a finite number of the U_λ already cover X.

4.6. If $\operatorname{Spec} A$ is disconnected then A contains an idempotent e (an element e satisfying $e^2=e$) distinct from 0 and 1.

4.7. If A and B are rings then $\operatorname{Spec}(A\times B)$ can be identified with the disjoint union $\operatorname{Spec} A \amalg \operatorname{Spec} B$, with both of these open and closed in $\operatorname{Spec}(A\times B)$.

4.8. If M is an A-module, N and N' submodules of M, and $S\subset A$ a multiplicative set, then $N_S \cap N'_S = (N\cap N')_S$, where both sides are considered as subsets of M_S.

4.9. A topological space is said to be *Noetherian* if the closed sets satisfy the descending chain condition. If A is a Noetherian ring then $\operatorname{Spec} A$ is a Noetherian topological space. (Note that the converse is not true in general.)

4.10. We say that a non-empty closed set V in a topological space is *reducible* if it can be expressed as a union $V=V_1\cup V_2$ of two strictly smaller closed sets V_1 and V_2, and *irreducible* if it does not have any such expression. If $\mathfrak{p}\in\operatorname{Spec} A$ then $V(\mathfrak{p})$ is an irreducible closed set, and conversely every irreducible closed set of $\operatorname{Spec} A$ can be written as $V(\mathfrak{p})$ for some $\mathfrak{p}\in\operatorname{Spec} A$.

4.11. Any closed subset of a Noetherian topological space can be written as a union of finitely many irreducible closed sets.

4.12. Use the results of the previous two exercises to prove the following: for I a proper ideal of a Noetherian ring, the set of prime ideals containing I has only finitely many minimal elements.