Math 8212 **Homework 4** **PJW**

Date due: Wednesday April 10, 2013. In class on Friday April 12 we will grade your answers, so it is important to be present on that day, with your homework.

The following 9 questions:

Questions 5.1, 5.2 on page 37 of Matsumura.

Questions 6.2, 6.3, 6.4, 6.5, 6.6 on page 42.

1. Find generators for the kernel of the ring homomorphism $\mathbb{Q}[X, Y, Z] \to \mathbb{C}$ specified by

$$X \mapsto \sqrt[3]{5}, \quad Y \mapsto e^{\frac{2\pi i}{3}}, \quad Z \mapsto (\sqrt[3]{5} + e^{\frac{2\pi i}{3}} + \sqrt{2}).$$

2. Suppose that the polynomial $f \in k[X_1, \ldots, X_n]$ where k is a field satisfies

$$f(\alpha_1, \ldots, \alpha_n) = 0 \quad \text{for some} \quad \alpha_1, \ldots, \alpha_n \in k.$$

Show that f lies in the ideal $(X - \alpha_1, \ldots, X - \alpha_n)$.

$a \in \mathfrak{p}_1 \dots \mathfrak{p}_{n-1}$ not belonging to $\mathfrak{p}_n$ and set $x = x' + ay$, this x satisfies our requirements. This proves the lemma.

Step 2. Setting $\sup\{b(\mathfrak{p}, M) | \mathfrak{p} \in \text{j-Spec}\, A\} = r$, we now show that there are just a finite number of primes $\mathfrak{p}$ such that $b(\mathfrak{p}, M) = r$. Indeed, for $n = 1, 2, \dots$, the subset $X_n = \{\mathfrak{p} \in \text{j-Spec}\, A | \mu(\mathfrak{p}, M) \geqslant n\}$ is closed in j-Spec A by Theorem 4.10; it has a finite number of irreducible components (by Ex. 4.11), and we let $\mathfrak{p}_{ni}$ (for $1 \leqslant i \leqslant \nu_n$) be their generic points. If M is generated by s elements then $X_n = \varnothing$ for $n > s$, so that the set $\{\mathfrak{p}_{nj}\}_{n,j}$ is finite. Let us prove that if $b(\mathfrak{p}, M) = r$ then $\mathfrak{p} \in \{\mathfrak{p}_{nj}\}_{n,j}$. Suppose $\mu(\mathfrak{p}, M) = n$; then $\mathfrak{p} \in X_n$, so that by construction $\mathfrak{p} \supset \mathfrak{p}_{ni}$ for some i. But if $\mathfrak{p} \neq \mathfrak{p}_{ni}$ then j-dim $\mathfrak{p} <$ j-dim $\mathfrak{p}_{ni}$, and since $\mu(\mathfrak{p}, M) = n = \mu(\mathfrak{p}_{ni}, M)$ we have $b(\mathfrak{p}, M) < b(\mathfrak{p}_{ni}, M)$, which is a contradiction. Hence $\mathfrak{p} = \mathfrak{p}_{ni}$.

Step 3. Let us choose an element $x \in M$ which is basic for each of the finitely many primes $\mathfrak{p}$ with $b(\mathfrak{p}, M) = r$, and set $\bar{M} = M/Ax$; then clearly $b(\mathfrak{p}, \bar{M}) \leqslant r - 1$ for every $\mathfrak{p} \in$ j-Spec A. Hence by induction $\bar{M}$ is generated by $r - 1$ elements, and therefore M by r elements. ■

Swan's paper contains a proof of the following generalisation to non-commutative rings: let A be a commutative ring, Λ a possibly non-commutative A-algebra and M a finite left Λ-module. Suppose that m-Spec A is Noetherian, and that for every maximal ideal $\mathfrak{p}$ of A the $\Lambda_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is generated by at most r elements; then M is generated as a Λ-module by at most $r + d$ elements, where d is the combinatorial dimension of m-Spec A.

The Forster–Swan theorem is a statement that local properties imply global ones; remarkable results in this direction have been obtained by Mohan Kumar [2] (see also Cowsik-Nori [1] and Eisenbud-Evans [1], [2]). The number of generators of ideals in local rings is the subject of a nice book by J. Sally [Sa].

Exercises to §5. Prove the following propositions.

5.1. Let k be a field, $R = k[X_1, \dots, X_n]$ and let $P \in \text{Spec}\, R$; then ht P + coht $P = n$.

5.2. A zero-dimensional Noetherian ring is Artinian (the converse to Example 2 above).

6 Associated primes and primary decomposition

Most readers will presumably have come across primary decomposition of ideals in Noetherian rings. This was the first big theorem obtained by Emmy Noether in her abstract treatment of commutative rings. Nowadays, as exemplified by Bourbaki [B4], the notion of associated prime is considered more important than primary decomposition itself.

$P_1 \in \mathrm{Ass}(M)$. The same works for the other P_i, and this proves that $\{P_1, \ldots, P_r\} \subset \mathrm{Ass}(M)$.

(iii) We have already seen that a proper submodule has an irreducible decomposition, so that by (i) it has a primary decomposition. Suppose that $N = N_1 \cap \cdots \cap N_r$ is a shortest primary decomposition, and that N_1 is the P-primary component with $P = P_1$. By Ex. 4.8 we know that $N_P = (N_1)_P \cap \cdots \cap (N_r)_P$, and for $i > 1$ a power of P_i is contained in $\mathrm{ann}(M/N_i)$; then since $P_i \not\subset P_1$ we have $(M/N_i)_P = 0$, and therefore $(N_i)_P = M_P$. Thus $N_P = (N_1)_P$, and hence $\varphi_P^{-1}(N_P) = \varphi_P^{-1}((N_1)_P)$; it is easy to check that the right-hand side is N_1. ■

Remark. The uniqueness of the P-primary component N, proved in (iii) for minimal primes P, does not hold in general; see Ex. 6.6.

Exercises to §6.

6.1. Find $\mathrm{Ass}(M)$ for the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})$.

6.2. If M is a finite module over a Noetherian ring A, and M_1, M_2 are submodules of M with $M = M_1 + M_2$ then can we say that $\mathrm{Ass}(M) = \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$?

6.3. Let A be a Noetherian ring and let $x \in A$ be an element which is neither a unit nor a zero-divisor; prove that the ideals xA and x^nA for $n = 1, 2\ldots$ have the same prime divisors:

$$\mathrm{Ass}_A(A/xA) = \mathrm{Ass}_A(A/x^nA).$$

6.4. Let I and J be ideals of a Noetherian ring A. Prove that if $JA_P \subset IA_P$ for every $P \in \mathrm{Ass}_A(A/I)$ then $J \subset I$.

6.5. Prove that the total ring of fractions of a reduced Noetherian ring A is a direct product of fields.

6.6. (Taken from [Nor 1], p. 30.) Let k be a field. Show that in $k[X, Y]$ we have

$$(X^2, XY) = (X) \cap (X^2, Y) = (X) \cap (X^2, XY, Y^2).$$

6.7. Let $f: A \longrightarrow B$ be a homomorphism of Noetherian rings, and M a finite B-module. Write ${}^af: \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ as in §4. Prove that ${}^af(\mathrm{Ass}_B(M)) = \mathrm{Ass}_A(M)$. (Consequently, $\mathrm{Ass}_A(M)$ is a finite set for such M.)

Appendix to §6. Secondary representations of a module

I.G. Macdonald [1] has developed the theory of attached prime ideals and secondary representations of a module, which is in a certain sense dual to the theory of associated prime ideals and primary decompositions. This theory was successfully applied to the theory of local cohomology by him and R.Y. Sharp (Macdonald & Sharp [1], Sharp [7]).

Let A be a commutative ring. An A-module M is said to be *secondary* if $M \neq 0$ and, for each $a \in A$, the endomorphism $\varphi_a: M \longrightarrow M$ defined