Math 8212 **Homework 5** **PJW**

Date due: Wednesday May 1, 2013. In class on Friday May 3 we will grade your answers, so it is important to be present on that day, with your homework.

Questions 8.1, 8.2 on page 63 of Matsumura.

In these questions p is a prime. We will write an element $a_{-s}p^{-s} + \cdots + a_{-1}p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots$ of the p-adic rationals $\mathbb{Q}_p$, where $0 \le a_i \le p-1$, as a string $\cdots a_3 a_2 a_1 a_0 . a_{-1} \cdots a_{-s}$ with a point separating a_0 and a_{-1}.

1. Show that when $p = 2$ we have

$$-1 = \cdots \overline{1}111. \quad \text{and}$$
$$\frac{1}{3} = \cdots \overline{10}101011.$$

   What fraction does $\cdots \overline{1100}1101.$ represent? Show that a p-adic integer is a negative (rational) integer if and only if it is of the form $\overline{1}a_n \cdots a_3 a_2 a_1 a_0$.

2. Show that $\mathbb{Q}$ is the subset of $\mathbb{Q}_p$ consisting of strings $\overline{a_m \cdots a_n} \cdots a_3 a_2 a_1 a_0 . a_{-1} \cdots a_{-s}$ which eventually recur, and that the localization $\mathbb{Z}_{(p)}$ is the subset of $\mathbb{Q}_p$ consisting of strings $\overline{a_m \cdots a_n} \cdots a_3 a_2 a_1 a_0.$ which eventually recur to the left and do not continue to the right of the point.

3. Write out a proof that the p-adic integers (described as strings of numbers as above) is isomorphic to the inverse limit of the diagram

$$\cdots \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

   in which each of the homomorphisms is a ring homomorphism (and hence is surjective).

4. Let $\mathcal{C}$ be a small category. Show that a sequence $F_1 \to F_2 \to F_3$ in the category $\mathrm{Fun}(\mathcal{C}, \mathrm{abgps})$ of functors from $\mathcal{C}$ to abelian groups, with natural transformations as morphisms, is exact if and only if for all objects X in $\mathcal{C}$ the sequence of abelian groups $F_1(X) \to F_2(X) \to F_3(X)$ is exact.

5. Write out a proof that if $\hat{M} = \varprojlim M/M_\lambda$ is the inverse limit, where the M_λ form a directed system as on page 55 of the book, then the topology on $\hat{M}$ given as the subspace topology of the product topology coincides with the linear topology given by the submodules M^*_λ which are the kernels of the projection maps $\hat{M} \to M/M_\lambda$.

6. Prove that the completion $\hat{M} = \varprojlim M/M_\lambda$ of the last question is itself complete with respect to the linear topology given by the submodules M^*_λ.

(3) The completion $\hat{A}$ of A is faithfully flat over A; hence $A \subset \hat{A}$, and $I\hat{A} \cap A = I$ for any ideal I of A.

(4) $\hat{A}$ is again a Noetherian local ring, with maximal ideal $\mathfrak{m}\hat{A}$, and it has the same residue class field as A; moreover, $\hat{A}/\mathfrak{m}^n\hat{A} = A/\mathfrak{m}^n$ for all $n > 0$.

(5) If A is a complete local ring, then for any ideal $I \neq A$, A/I is again a complete local ring.

Remark 1. Even if A is complete, the localisation $A_\mathfrak{p}$ of A at a prime $\mathfrak{p}$ may not be.

Remark 2. An Artinian local ring $(A, \mathfrak{m})$ is complete; in fact, it is clear from the proof of Theorem 3.2 that there exists a ν such that $\mathfrak{m}^\nu = 0$, so that $\hat{A} = \varprojlim A/\mathfrak{m}^n = A$.

Exercises to §8. Prove the following propositions.

8.1. If A is a Noetherian ring, I and J are ideals of A, and A is complete both for the I-adic and J-adic topologies, then A is also complete for the $(I+J)$-adic topology.

8.2. Let A be a Noetherian ring, and $I \supset J$ ideals of A; if A is I-adically complete, it is also J-adically complete.

8.3. Let A be a Zariski ring and $\hat{A}$ its completion. If $\mathfrak{a} \subset A$ is an ideal such that $\mathfrak{a}\hat{A}$ is principal, then $\mathfrak{a}$ is principal.

8.4. According to Theorem 8.12, if $y \in \bigcap_\nu I^\nu$ then

$$y \in \sum_{i=1}^{n} (X_i - a_i)A[[X_1, \ldots, X_n]].$$

Verify this directly in the special case $I = eA$, where $e^2 = e$.

8.5. Let A be a Noetherian ring and I a proper ideal of A; consider the multiplicative set $S = 1 + I$ as in §4, Example 3. Then A_S is a Zariski ring with ideal of definition IA_S, and its completion coincides with the I-adic completion of A.

8.6. If A is I-adically complete then $B = A[[X]]$ is $(IB + XB)$-adically complete.

8.7. Let $(A, \mathfrak{m})$ be a complete Noetherian local ring, and $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ a chain of ideals of A for which $\bigcap_\nu \mathfrak{a}_\nu = (0)$; then for each n there exists $\nu(n)$ for which $\mathfrak{a}_{\nu(n)} \subset \mathfrak{m}^n$. In other words, the linear topology defined by $\{\mathfrak{a}_\nu\}_{\nu=1,2,\ldots}$ is stronger than or equal to the $\mathfrak{m}$-adic topology (Chevalley's theorem).

8.8. Let A be a Noetherian ring, $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ ideals of A; if M is a finite A-module and $N \subset M$ a submodule, then there exists $c > 0$ such that

$$n_1 \geqslant c, \ldots, n_r \geqslant c \Rightarrow \mathfrak{a}_1^{n_1} \ldots \mathfrak{a}_r^{n_r} M \cap N = \mathfrak{a}_1^{n_1 - c} \ldots \mathfrak{a}_r^{n_r - c}(\mathfrak{a}_1^c \ldots \mathfrak{a}_r^c M \cap N).$$

8.9. Let A be a Noetherian ring and $P \in \mathrm{Ass}(A)$. Then there is an integer $c > 0$ such that $P \in \mathrm{Ass}(A/I)$ for every ideal $I \subset P^c$ (hint: localise at P).

8.10. Show by example that the conclusion of Ex. 8.7. does not necessarily hold if A is not complete.