Triangulated and Derived Categories

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1. Categories of complexes

An additive category $\mathcal{A}$ is one where $\mathrm{Hom}(X, Y)$ is an abelian group such that composition is bilinear. Direct sums exist in $\mathcal{A}$. These are supposed to be biproducts, i.e. simultaneously products and coproducts. For example, apart from taking $\mathcal{A}$ to be R-mod for some ring R, we could take it to be the full subcategory of projective modules, or of injective modules.

$C(\mathcal{A})$ is the category of chain complexes

$$X: \qquad \cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

We have a shift operator

$$X[1]: \qquad \cdots \to X_n \xrightarrow{-d_n} X_{n-1} \xrightarrow{-d_{n-1}} X_{n-2} \to \cdots$$

so that $X[1]_n = X_{n-1}$. A *stalk complex* is one which is non-zero in only one degree. We call a complex *contractible* if it is chain homotopy equivalent to the zero complex. Given a complex X consider the following two maps of complexes:

$$\begin{array}{ccccccccc}
X: & \cdots & \to & X_2 & \xrightarrow{d} & X_1 & \xrightarrow{d} & X_0 & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle \iota_X} & & & \downarrow{\scriptstyle \begin{pmatrix} d \\ 1 \end{pmatrix}} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \downarrow{\scriptstyle \begin{pmatrix} d \\ 1 \end{pmatrix}} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \downarrow{\scriptstyle \begin{pmatrix} d \\ 1 \end{pmatrix}} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\
I_X: & \cdots & \to & X_1 \oplus X_2 & \longrightarrow & X_0 \oplus X_1 & \longrightarrow & X_{-1} \oplus X_0 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \pi_X} & & & \downarrow{\scriptstyle \begin{pmatrix} 1 & -d \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & -d \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & -d \end{pmatrix}} & & \\
X[1]: & \cdots & \to & X_1 & \xrightarrow{-d} & X_0 & \xrightarrow{-d} & X_{-1} & \xrightarrow{-d} & \cdots
\end{array}$$

We see that the bottom complex is $X[1]$, and that if these are complexes of modules then each vertical sequence is a short exact sequence. In fact the diagram is a short exact sequence of complexes.

Proposition 1.1. (1) *A complex is contractible if and only if it is the direct sum of complexes of the form* $\cdots \to 0 \to M \xrightarrow{1} M \to 0 \to \cdots$

(2) *A chain map $X \to Y$ is chain homotopic to 0 if and only if it can be factored through a contractible complex, if and only if it can be factored through ι_X.*

We see from statement (1) of the above proposition that the complex I_X is contractible.

The complexes of the proposition are projective and injective relative to a certain structure giving an *exact category* in the sense of Quillen. For this we take an additive category $\mathcal{A}$ together with a class $\mathcal{S}$ of so-called *admissible short exact sequences* which are required to satisfy the following axioms: if the sequences look like $X \xrightarrow{i} Y \xrightarrow{s} Z$ we call i an *admissible mono* and s an *admissible epi* and require that

(1) $i = \operatorname{Ker} s$ and $s = \operatorname{Coker} i$ (this implies that i must be mono and s must be epi),
(2) isomorphisms are admissible monomorphisms
(3) the composite of admissible monomorphisms is an admissible monomorphisms,
(4) pushouts

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{i'} & Y' \end{array}$$

exist whenever i is an admissible monomorphism, and then i' is also an admissible monomorphism.

We require also that the dual statements to these hold.

As examples we might take $\mathcal{S}$ to consist of all short exact sequences (if that makes sense in $\mathcal{A}$), or all split short exact sequences. When dealing with complexes of modules, we will take $\mathcal{S}$ to be all short exact sequences of complexes which are split in each degree. Such sequences need not split as sequences of complexes: for example, if X is a stalk complex the sequence $X \to I_X \to X[1]$ is seen not to be split, because I_X has zero homology.

We say that X is projective or injective relative to $\mathcal{S}$ according as $\operatorname{Hom}(X, -)$ or $\operatorname{Hom}(-X,)$ is exact on sequences in $\mathcal{S}$.

Proposition 1.2. *The following are equivalent for an object X in an exact category:*

(1) *X is $\mathcal{S}$-projective,*
(2) *X satisfies the projective lifting property with respect to admissible epimorphisms,*
(3) *all exact sequences $0 \to V \to W \to X \to 0$ in $\mathcal{S}$ split.*

Proposition 1.3. *Let $\mathcal{S}$ in $C(R\text{-}mod)$ be the exact sequences of complexes which are split in each degree. The injectives and projectives relative to S coincide, and are the contractible complexes.*

Corollary 1.4. *$C(R\text{-}mod)$ has enough projectives and injectives relative to $\mathcal{S}$.*

We say that an exact category with admissible short exact sequences $\mathcal{S}$ is a *Frobenius category* if it has enough projectives and injectives relative to $\mathcal{S}$, and the projectives and injectives coincide.

Corollary 1.5. *With the exact structure given by sequences which are split in each degree, $C(R\text{-}mod)$ is a Frobenius category.*

Another example of a Frobenius category is provided by kG-mod where G is a finite group and k is a field. The group algebra is self-injective, meaning that injectives and projectives coincide.

Given a Frobenius category $\mathcal{A}$ we define the stable category $\underline{\mathcal{A}}$ to have the same objects as $\mathcal{A}$, and $\operatorname{Hom}_{\underline{\mathcal{A}}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(X, Y)/\sim = \operatorname{Hom}(X, Y)/I(X, Y)$ where

$\alpha \sim \beta$ if and only if $\alpha - \beta$ factors through a projective (or injective) and $I(X,Y)$ is the subgroup consisting of homomorphisms which factor through a projective.

Starting with kG-mod we get the stable module category for kG, and starting with $C(R\text{-mod})$ we get the homotopy category of complexes of R-modules $K(R\text{-mod})$.

Proposition 1.6. *Every morphism in $K(R\text{-mod})$ can be represented by an admissible monomorphism.*

Proof. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \| & & \uparrow{\scriptstyle \pi_Y} \\ X & \longrightarrow & I_X \oplus Y \end{array}$$

In the homotopy category π_Y is an isomorphism since $1_{I_X \oplus Y} - \iota_Y \pi_Y$ factors through I_X. $\square$

When we come to the octahedral axiom for triangulated categories it will be helpful to observe that in an exact category the third isomorphism theorem makes sense and is true. For modules, this isomorphism theorem says that if $L \subseteq M \subseteq N \subseteq$ then $(N/L)/(M/L) \cong N/M$. We can express this as a diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & L & \longrightarrow & M & \longrightarrow & M/L & \to & 0 \\ & & \| & & \downarrow & & \exists\downarrow & & \\ 0 & \to & L & \longrightarrow & N & \longrightarrow & N/L & \to & 0 \\ & & & & \downarrow & & \exists\downarrow & & \\ & & & & N/M & = & N/M & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

2. Triangulated Categories

A *triangulated category* is an additive category $\mathcal{T}$ with an automorphism $\mathcal{T} \to \mathcal{T}$ written $X \mapsto X[1]$ satisfying certain axioms. In our examples when $\mathcal{A}$ is a Frobenius category the automorphism will be constructed via the sequence $X \to I_X \to X[1]$, which is well-defined up to natural isomorphism in $\underline{\mathcal{A}}$. In $C(R\text{-Mod})$ this construction gives the usual shift of complexes. We require $\mathcal{T}$ to have a class of diagrams $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ called *triangles*, or *distinguished triangles.*

Starting from a Frobenius category the triangles may be taken to be constructed from commutative diagrams

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, equal] & Y \arrow[r, "\beta"] \arrow[d] & Z \arrow[d, "\gamma"] \\
X \arrow[r] & I_X \arrow[r] & X[1]
\end{tikzcd}$$

together with the diagrams isomorphic to these. The axioms are:

TR1 Always $X \xrightarrow{1} X \to 0 \to X[1]$ is a triangle; for every morphism $X \xrightarrow{\alpha} Y$ there exists a triangle $X \xrightarrow{\alpha} Y \to Z \to X[1]$; triangles are closed under isomorphism.

TR2 $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ is a triangle if and only if $Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1] \xrightarrow{-\alpha[1]} Y[1]$ is a triangle.

TR3 Given a diagram of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

in which the left hand square commutes, there exists a morphism $Z \to Z'$ making the whole diagram commute. (The morphism need not be unique.)

TR4 (octahedral axiom) Given three triangles and a commutative diagram

$$\begin{tikzcd}
& Z[-1] \arrow[r, equal] \arrow[d] & Z[-1] \arrow[d, "\exists"'] & \\
U \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "\exists"'] & U[1] \arrow[d, equal] \\
U \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "\exists"'] & U[1] \\
& Z \arrow[r, equal] & Z &
\end{tikzcd}$$

there exists a fourth triangle so that the diagram commutes

Proposition 2.1. *The triangles coming from a Frobenius category as above satisfy the axioms.*

Proposition 2.2. *Suppose that $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ is a triangle in a triangulated category $\mathcal{T}$. Then*

(1) $\beta\alpha = 0$.

(2) *For any object W we obtain a long exact sequences*

$$\mathrm{Hom}(W,X) \xrightarrow{\alpha_*} \mathrm{Hom}(W,Y) \xrightarrow{\beta_*} \mathrm{Hom}(W,Z) \xrightarrow{\gamma_*} \mathrm{Hom}(W,X[1]) \xrightarrow{\alpha[1]_*} \cdots$$

and

$$\mathrm{Hom}(X,W) \xleftarrow{\alpha^*} \mathrm{Hom}(Y,W) \xleftarrow{\beta^*} \mathrm{Hom}(Z,W) \xleftarrow{\gamma^*} \mathrm{Hom}(X[1],W) \xleftarrow{\alpha[1]^*} \cdots$$

(We say that $\mathrm{Hom}(X,-)$ *and* $\mathrm{Hom}(-,X)$ *are cohomological functors.)*

(3) *Given a commutative diagram of triangles*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\theta[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

if θ and ϕ are isomorphisms then so is ψ.

(4) *There is only one triangle starting with α, up to isomorphism.*

(5) *$Z \cong 0$ if and only if α is an isomorphism.*

(6) *$\gamma = 0 \Leftrightarrow \beta$ is split epi $\Leftrightarrow \beta$ is epi $\Leftrightarrow \alpha$ is split mono $\Leftrightarrow \alpha$ is mono*

3. The Mapping Cone

Given a morphism of complexes $f : X \to Y$ we define the mapping cone $C(f)$ of f to be the total complex of the double complex

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d^X_{n+2}} & X_{n+1} & \xrightarrow{d^X_{n+1}} & X_n & \xrightarrow{d^X_n} & X_{n-1} & \xrightarrow{d^X_{n-1}} & \cdots \\ & & {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & \\ \cdots & \xrightarrow{d^Y_{n+2}} & Y_{n+1} & \xrightarrow{d^Y_{n+1}} & Y_n & \xrightarrow{d^Y_n} & Y_{n-1} & \xrightarrow{d^Y_{n-1}} & \cdots \end{array}$$

Explicitly, this is the complex

$$C(f): \quad \cdots \to X_n \oplus Y_{n+1} \xrightarrow{\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}} X_{n-1} \oplus Y_n \xrightarrow{\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}} X_{n-2} \oplus Y_{n-1} \xrightarrow{\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}} \cdots$$

As an example of this, the mapping cone of $1_X : X \to X$ may be identified as the complex I_X previously constructed. To see this, rewrite

$$C_n(1_X) = X_{n-1} \oplus \{\begin{pmatrix} -dx \\ x \end{pmatrix} \mid x \in X_n\}$$

The second term on the right is isomorphic to X_n, and with respect to this decomposition the matrix of the boundary map is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We have also been constructing the mapping cone in our description of triangles in the homotopy category of complexes.

Proposition 3.1. *Let $f : X \to Y$ be a morphism of chain complexes.*

(1) *There is a short exact sequence of complexes $0 \to Y \to C(f) \to X[1] \to 0$, and hence a long exact sequence of homology groups*

$$\cdots \to H_n(Y) \to H_n(C(f)) \to H_{n-1}(X) \to H_{n-1}(Y) \to \cdots$$

(2) *$C(f)$ is acyclic if and only if f induces an isomorphism on homology.*

(3) *There is a short exact sequence of complexes*

$$0 \to X \xrightarrow{\binom{f}{\iota}} Y \oplus I_X \to C(f) \to 0.$$

Thus the triangle determined by f in $K(R\text{-mod})$ has the form

$$X \xrightarrow{f} Y \to C(f) \to X[1].$$

(4) *$C(f)$ is contractible if and only if f is a chain homotopy equivalence.*

The last part of the above proposition can be proved more directly, without exploiting our excursion into triangulated categories.

As an example we can construct triangles starting with module homomorphisms. If $f : U \to V$ is a homomorphism of R-modules we may regard these as complexes concentrated in degree 0 and we obtain a triangle

$$U \xrightarrow{f} V \to \begin{pmatrix} U \\ \downarrow \\ V \end{pmatrix} \to U[1].$$

In case f is a monomorphism it is tempting to identify the third term in the triangle with $\operatorname{Coker} f$, or if f is an epimorphism we might think to identify the third term with $\operatorname{Ker} f[1]$, but in the homotopy category of complexes we cannot do this. In the derived category we will make such an identification, and this is one reason for introducing the derived category. Observe that if f is a monomorphism we have a morphism of complexes

$$\begin{pmatrix} U \\ \downarrow \\ V \end{pmatrix} \to \operatorname{Coker} f$$

which is an isomorphism on homology, but is not a homotopy equivalence.

4. Projective resolutions

We say that a map of complexes $f : X \to Y$ is a homology isomorphism, or *quasi-isomorphism* if it induces an isomorphism of homology groups. We define a *projective resolution* of a complex X to be a quasi-isomorphism $P \to X$ where P is a complex of projective modules. An *injective resolution* is a quasi-isomorphism $X \to I$ where I is a complex of injective modules. When X is zero except in degree zero these notions coincide with the usual definitions of projective and injective resolutions of a module.

We say that a complex X is *bounded below* if there exists N so that X is zero in all degrees below N. Write $C^+(R\text{–Mod})$ and $K^+(R\text{–Mod})$ for the full subcategories of $C(R\text{–Mod})$ and $K(R\text{–Mod})$ whose objects are the complexes which are bounded below. Similarly X is *bounded above* if there exists N so that X is zero in all degrees above N and these are the objects of $C^-(R\text{–Mod})$ and $K^-(R\text{–Mod})$. We say X is *bounded* if it is both bounded below and bounded above and we write $C^b(R\text{–Mod})$ and $K^b(R\text{–Mod})$ for the full subcategories whose objects are the bounded complexes. Let R−Proj denote the full subcategory of R−Mod whose objects are the projective R-modules (together with 0). For each $\epsilon \in \{\emptyset, +, -, b\}$ we have subcategories $C^\epsilon(R\text{−Proj})$ of $C^\epsilon(R\text{−Mod})$ which are Frobenius categories with admissible exact sequences the sequences of appropriately bounded complexes of projective modules which split in each degree. This gives rise to stable categories $K^\epsilon(R\text{−Proj})$ which are triangulated. We will also be interested in the triangulated category $K^{+,b}(R\text{−Proj})$ formed from $C^{+,b}(R\text{−Proj})$ whose objects are the complexes of projective modules which are bounded below, and in which the homology is zero above some dimension.

Proposition 4.1. *Any quasi isomorphism in $K^+(R\text{−Proj})$ is an isomorphism.*

Proof. If $F : P \to Q$ is a quasi-isomorphism of complexes of projective modules bounded below then $C(f)$ is acyclic, and a complex of projective modules, bounded below. It follows that $C(f)$ is contractible. Hence f is a homotopy equivalence. □

Proposition 4.2. *Any complex of modules X which is bounded below has a projective resolution.*

5. The Derived Category

The derived categories $D^\epsilon(R\text{−Mod})$ where $\epsilon \in \{\emptyset, +, -, b\}$ are constructed from the categories $K^\epsilon(R\text{−Mod})$ by requiring them to have the same objects, and morphisms which are obtained from $K^\epsilon(R\text{−Mod})$ by formally inverting all quasi-isomorphisms. It turns out that the derived category is an additive category and the triangles in $K^\epsilon(R\text{−Mod})$ (together with triangles isomorphic to them) become triangles in $D^\epsilon(R\text{−Mod})$ making the derived category a triangulated category. There is a functor of triangulated categories $K^\epsilon(R\text{−Mod}) \to D^\epsilon(R\text{−Mod})$ for each ϵ which is universal among functors $K^\epsilon(R\text{−Mod}) \to \mathcal{A}$ which send quasi-isomorphism to isomorphisms.

We considered before the triangle

$$U \xrightarrow{f} V \to \begin{pmatrix} U \\ \downarrow \\ V \end{pmatrix} \to U[1].$$

If f is a monomorphism the quasi-isomorphism

$$\begin{pmatrix} U \\ \downarrow \\ V \end{pmatrix} \to \operatorname{Coker} f$$

becomes an isomorphism in the derived category. From this we see that any short exact sequence of modules $0 \to U \to V \to W \to 0$ gives rise to a triangle in the derived category $U \to V \to W \to U[1]$. We will see later that the map $W \to U[1]$ may be interpreted as specifying the Ext class of the extension.

Theorem 5.1. *The functor $K^+(R\text{−Mod}) \to D^+(R\text{−Mod})$ restricts to a functor $K^+(R\text{−Proj}) \to D^+(R\text{−Mod})$ and also to a functor $K^{+,b}(R\text{−Proj}) \to D^b(R\text{−Mod})$ which are equivalences of categories.*

We might expect this to be true because every object in $D^+(R-\mathrm{Mod})$ has a projective resolution and so is isomorphic to an object of $K^+(R-\mathrm{Proj})$. Also every quasi-isomorphism in $K^+(R-\mathrm{Proj})$ is invertible, and so we do not need to invert any further morphisms in passing to $D^+(R-\mathrm{Mod})$.

The consequence of this is that to compute morphisms in $D^+(R-\mathrm{Mod})$ between two objects X and Y we may replace X and Y by projective resolutions P_X and P_Y and then computing $\mathrm{Hom}_{K^+(R-\mathrm{Proj})}(P_X, P_Y)$. In fact we can do better than this: we only need to replace X by its projective resolution.

Proposition 5.2. *Let X and Y be complexes of R-modules which are bounded below. Then* $\mathrm{Hom}_{D^+(R-\mathrm{Mod})}(X, Y) \cong \mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P_X, Y)$.

Proof. We know that

$$\mathrm{Hom}_{D^+(R-\mathrm{Mod})}(X, Y) \cong \mathrm{Hom}_{K^+(R-\mathrm{Proj})}(P_X, P_Y) = \mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P_X, P_Y).$$

We have a quasi-isomorphism $f : P_Y \to Y$ with mapping cone $C(f)$ which is acyclic. Any map of complexes $P_X \to C(f)$ is homotopic to zero. Hence from the long exact sequence obtained by applying $\mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P_X, -)$ to the triangle $P_Y \to Y \to C(f) \to P_Y[1]$ we see that

$$\mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P_X, P_Y) \cong \mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P_X, Y).$$

□

Proposition 5.3. *Let M and N be R-modules, which we regard as complexes concentrated in degree zero. Then*

$$\mathrm{Hom}_{D(R-\mathrm{Mod})}(M, N[t]) \cong \begin{cases} \mathrm{Ext}^t_R(M, N) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Proof. We calculate this homomorphism group by taking a projective resolution P of M and computing $\mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P, N[t])$ by considering the diagram

$$\begin{array}{ccccc} P_{t+1} & \xrightarrow{d_{t+1}} & P_t & \xrightarrow{d_t} & P_{t-1} \\ \downarrow{\scriptstyle f_{t+1}} & & \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f_{t-1}} \\ 0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Evidently these homomorphisms are zero if $t < 0$. If $t \geq 0$, write B_t for the image of d_t. Since $f_t d_{t+1} = 0$ we see that f_t is zero on B_{t+1} so factors through a map $B_t \cong P_t/B_{t+1} \to N$. It is homotopic to zero if and only if there is a map $T_{t-1} : P_{t-1} \to N$ so that $T_{t-1} d_t = f_t$, or in other words if the map $B_t \to N$ is the restriction of a map from P_{t-1}. Thus $\mathrm{Hom}_{K^+(R-\mathrm{Mod})}(P, N[t])$ identifies as the cokernel of $\mathrm{Hom}(P_{t-1}, N) \to \mathrm{Hom}(B_t, N)$. This cokernel also computes $\mathrm{Ext}^T_R(M, N)$. □

6. Hereditary algebras

One goal is to describe in detail the derived category in the case of *hereditary algebras*, namely algebras for which submodules of projective modules are always projective. As preparation for this we collect some results about hereditary algebras.

Proposition 6.1. *Let R be a finite dimensional hereditary algebra over a field k. Every indecomposable object of $D^+(R-mod)$ or of $D^b(R-mod)$ is isomorphic to the shift of a module.*

Proof. Take an indecomposable complex of projective modules P. We may assume that it is zero in negative degrees and that it has non-zero zero homology $H_0(P) = M$. Take a minimal projective resolution Q of M. By lifting the identity on M from P to Q and also from Q to P we get maps whose composition on Q is an isomorphism by Nakayama's lemma, and because Q at most two non-zero terms. From this we deduce that Q is a summand of P, and hence that $Q = P$ since P is indecomposable. This Q is isomorphic in the derived category to M. $\square$

Exercise: Prove the converse of the above result.

The hereditary algebras we will consider are path algebras of quivers. A *quiver* Q is a directed graph, and a *path* in it is a list of (directed) edges so that the end point of each edge is the starting point of the next edge. We include by convention for each vertex the empty path starting and finishing at that vertex. The *path algebra* kQ is the vector space with the set of paths as a basis and with multiplication of basis elements being concatenation of paths where the end of one is the start of the other, and otherwise zero.

For example if the quiver Q is $\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ}$ there are six paths $e_{i,j}, j \leq i$ where $e_{i,j}$ is the path going from vertex j to vertex i. In the path algebra these multiply as $e_{i,j}e_{k,l} = e_{i,l}$ if $j = k$, 0 otherwise, and this is the same as the multiplication of the 3×3 matrices $E_{i,j}$ which are 1 in position (i, j) and 0 elsewhere. From this we see that the path algebra of this quiver is isomorphic to the algebra R of lower triangular 3×3-matrices. This algebra has up to isomorphism 3 simple modules S_1, S_2, S_3 each of dimension 1, associated to the three vertices of the quiver. Their projective covers P_1, P_2, P_3 are the three modules which appear as the spaces of column vectors in the algebra of lower triangular matrices, so that $R \cong P_1 \oplus P_2 \oplus P_3$ as left R-modules. Each of these indecomposable projective modules has a unique composition series, and we have monomorphisms $P_3 \to P_2 \to P_1$ whose images in P_1 form the unique composition series of P_1 with composition factors S_3, S_2, S_1 starting from the bottom. The 6 indecomposable R-modules are conveniently listed in the Auslander-Reiten quiver of R.

7. Auslander-Reiten triangles

For this we follow Chapter 1 Section 4 of Happel's book.

We will assume that $\mathcal{C}$ is a triangulated category such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a finite dimensional vector space over k for all objects X and Y, and that the endomorphism ring of each indecomposable object is local. It follows that the Krull-Schmidt theorem holds in $\mathcal{C}$.

etc.

8. Computation of the bounded derived category for hereditary algebras

For this we follow Chapter 1 Section 5 of Happel's book.

9. Tilting theory

10. The cluster category

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