Math 8246 **Homework 1** **PJW**

Date due: Monday February 15, 2016. We will discuss these questions on Wednesday 2/17/2016

1. Show that the two extensions $\mathbb{Z}\xrightarrow{\mu}\mathbb{Z}\xrightarrow{\epsilon}\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}\xrightarrow{\mu'}\mathbb{Z}\xrightarrow{\epsilon'}\mathbb{Z}/3\mathbb{Z}$ are not equivalent, where $\mu = \mu'$ is multiplication by 3, $\epsilon(1) \equiv 1 \pmod 3$ and $\epsilon'(1) \equiv 2 \pmod 3$.

2. Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of $\mathbb{Z}G$-modules, then for every $n \geq 0$ the sequence $0 \to H^n(G, L) \to H^n(G, M) \to H^n(G, N) \to 0$ is also short exact and split. [Use a splitting homomorphism and functoriality of H^n.]

3. (a) Let M and N be $\mathbb{Z}G$-modules and suppose that N has the trivial G-action. Show that $\mathrm{Hom}_{\mathbb{Z}G}(M, N) \cong \mathrm{Hom}_{\mathbb{Z}G}(M/(IG \cdot M), N)$.
(b) Show that for all groups G, $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, IG) = 0$; and that if we suppose that G is finite then $\mathrm{Hom}_{\mathbb{Z}G}(IG, \mathbb{Z}) = 0$.
(c) By applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(IG, \quad)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ show that for all finite groups G, if $f : IG \to \mathbb{Z}G$ is any $\mathbb{Z}G$-module homomorphism then $f(IG) \subseteq IG$.
(d) Show that if G is finite and $d : G \to \mathbb{Z}G$ is any derivation then $d(G) \subseteq IG$. Is the same true for arbitrary groups G?

4. Let G be a finite group. Show that the endomorphism ring $\mathrm{Hom}_{\mathbb{Z}G}(IG, IG)$ is isomorphic to $\mathbb{Z}G/(N)$ where $N = \sum_{g\in G} g$ is the norm element which generates $(N) = (\mathbb{Z}G)^G$.
[You may assume that every $\mathbb{Z}G$-module homomorphism $IG \to \mathbb{Z}G$ has image contained in IG. Apply the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z}G)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$. You may assume for a finite group G that $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$.]

5. Show that for every group G:
(a) all derivations $d : G \to M$ satisfy $d(1) = 0$, and
(b) the mapping $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$ is a derivation.

6. (a) Show that the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ is split as a sequence of $\mathbb{Z}G$-modules if and only if $G = 1$. Deduce that the identity group is the only group of cohomological dimension 0.
(b) Show that if G is a free group then $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) \neq 0$.

7. Suppose that we have two commutative diagrams of group homomorphisms

$$\begin{array}{ccccccccc}
1 & \to & L & \xrightarrow{\gamma} & J & \to & G & \to & 1 \\
 & & \downarrow\theta & & \downarrow\phi_i & & \downarrow 1_G & & \\
1 & \to & M & \xrightarrow{\alpha_i} & E_i & \to & G & \to & 1
\end{array}$$

where $i = 1, 2$, the maps labeled without the suffix i are the same in both diagrams, L and M are abelian and the two rows are group extensions (i.e. short exact sequences of groups). Assume that the two module actions of G on M given by conjugation within E_1 and E_2 are the same. Show that the two bottom extensions are equivalent. [Hint: one way to proceed is to show that they are both equivalent to a third extension which you construct.]

Extra questions: do not hand in!

8. If N is a right $\mathbb{Z}G$-module and M is a left $\mathbb{Z}G$-module we may make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = ng^{-1} \otimes gm$, extended linearly to the whole of $N \otimes_{\mathbb{Z}} M$. Show that $N \otimes_{\mathbb{Z}G} M \cong (N \otimes_{\mathbb{Z}} M)_G$.
[Not part of the question, just information: if N and M are two left modules we make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = gn \otimes gm$. This is called the *diagonal action* on the tensor product.]

9. Let $0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0$ be a short exact sequence.
   (i) Construct its inverse under the group operation in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ with sufficient precision that you can determine by examination of the two sequences whether or not they are equivalent.
   (ii) Determine the isomorphism type of middle term of the sum of the sequence with itself. [By 'the sum' is meant the addition in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$.]

10. Let $G = \langle g \rangle$ be an infinite cyclic group. Consider an extension of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z} \to 0$$

in which the maps are inclusion into the first summand and projection onto the second summand, and where g acts on $\mathbb{Z} \oplus \mathbb{Z}$ as the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with respect to the basis given by this direct sum decomposition. In the identification $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, determine the Ext class of this extension, and conclude that the extension is not split. Find a description of an extension represented by $5 \in \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$.