Some review questions for exam 1. The purpose of this sheet is to assist you by providing some more questions that may be helpful. It is not supposed to give an indication of what the exam questions will be like, or the distribution of topics, or anything like that.

Many questions in the book are appropriate review. There are standard things to do with matrices, and with lists of vectors: are they linearly independent, find the dimension of the space they span, find a basis for the space they span. For example, look at 2.3.21-2.3.24; 2.5.21- 2.5.25. These questions include finding a basis for the kernel of a matrix, that kind of thing.

There are theoretical questions, such as 2.5.33 - 2.5.46. There are questions to do with abstract vector spaces, such as 2.5.27, 2.4.10 - 2.4.27. From chapter 1 there are questions about determinants in section 1.9, and about finding factorizations of matrices, such as question 1.5.32.

Here are more questions:

1. Let C([0, 1]) be the vector space consisting of real valued functions which are defined and continuous on the interval [0, 1]. Do the following conditions define subspaces of C([0, 1]) respectively? Justify your answer.

(a) $\int_0^t f(s)\sin(s)ds = \sin(t)$ for all t in $[0, 1]$;

(b) $f(1-t) = f(t)$ for all t in $[0, 1]$.

2. Let A be the matrix $\begin{pmatrix} 2 & 3 & 7 & 0 \\ 1 & 2 & 5 & -1 \\ 2 & 1 & 1 & 4 \end{pmatrix}$ Find bases for its kernel, range, cokernel and corange.

3. Let $V = M_{2,2}$ be the vector space of all 2 by 2 matrices with real coefficients with usual addition and scalar multiplication of matrices. Let H be the subset defined by $H = \{\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a$ and b are in $\mathbb{R}\}$. Show that H is a subspace of V.

4. Let A be the matrix $A = \begin{pmatrix} 1 & -3 & 1 & 0 \\ 1 & 0 & 5 & 1 \\ -2 & 9 & 9 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$.

(a) Compute the determinant of A by any method you wish. Show your work.

(b) If B and C are both unknown 4×4 matrices with $\det(B) = 2$ and $\det(C) = 9$, what is the determinant of the product $C^{-1}AB$? Briefly explain.

5. Consider the following collection of four polynomials in $P^{(2)}$: $p_1(x) = -1 + x - 2x^2$, $p_2(x) = -3 + 5x - 2x^2$, $p_3(x) = -1 - 4x^2$, $p_4(x) = -2 + 2x - 8x^2$.

(a) Do these four polynomials span $P^{(2)}$? Fully justify your answer.

(b) Find a subset of these four polynomials which forms a basis for $P^{(2)}$ and explain why it is a basis.

6. For each statement below, determine whether it is True or False. If it is true, briefly explain why it is true. If it is false, give a specific example or briefly show that it is false.

(a) If A, B are two square matrices and AB is invertible, then BA is also invertible.

(b) The columns of a matrix are linearly independent if and only if its rank equals the number of columns.

(c) The set of functions $f \in \mathcal{F}([-10, 10])$ satisfying $f(2) = 3f(-3)$ is a subspace of the function space $\mathcal{F}([-10, 10])$.

(d) A linear system of two equations in three unknowns must have more than one solution.

(e) If A is a regular $n \times n$ matrix, then for any $b \in \mathbb{R}^n$, the linear system $Ax = b$ has exactly one equation.

(f) If A is an invertible $n \times n$ matrix, then the matrix equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

(g) Suppose that v_1, v_2, v_3, w are vectors in a vector space V which satisfy the equations $-8v_1 + v_2 = w$ and $2v_1 + 4v_2 - v_3 = w$. Then the vectors v_1, v_2, v_3 must be linearly dependent.

(h) The nullspace (kernel) of a 4×6 matrix C may be 1-dimensional.

(i) If $A = BC$ is invertible then B and C are also invertible.

(j) If $A = BC$ is invertible and B, C are square then B and C are also invertible.

(k) If A is square and has a right inverse, then A is invertible.

7. Give examples of matrices with the following properties, or give a short explanation of why it is impossible:

(a) An 1×2 matrix A for which $\ker(A)$ only contains the zero vector.

(b) Three linearly independent 1×2 matrices A, B and C (in the vector space of all 1×2 matrices).

8. (a) Let $P^{(4)}$ be the vector space of all polynomials in x of degree less or equal to 4. What is the dimension of the subspace spanned by $x^2 + x + 1$, $x^2 - x$ and $5x^2 - x + 2$?

(b) Let A, B be two matrices so that their product AB is defined. Show the range of AB is contained in the range of A.

9. Find the general solution of $\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

10. Find a basis for the subspace of $P^{(3)}$ consisting of polynomials $p(x)$ such that $p(1) = 0$.