Math 8300 **Homework 1** **PJW**

Date due: Monday January 30, 2017. We will discuss these questions on Wednesday 2/1/2017

1. (2 pts) Let M be a kG-module. Show that M admits a non-singular G-invariant bilinear form if and only if $M \cong M^*$ as kG-modules.

2. Let M be a kG-module and let $\mathcal{B}$ be the vector space of bilinear forms $M \times M \to k$.
   a) (2 pts) For each $g \in G$ we may construct two new bilinear forms $\langle -, - \rangle_1^g : v, w \mapsto \langle vg, wg \rangle$, and $\langle -, - \rangle_2^g : v, w \mapsto \langle vg^{-1}, wg^{-1} \rangle$. One of these definitions makes $\mathcal{B}$ into a kG-module via $\langle -, - \rangle \cdot g = \langle -, - \rangle_i^g$, $i = 1$ or 2. Which value of i achieves this?
   b) (0 pts) We note without further comment that a bilinear form is G-invariant $\Leftrightarrow$ it is fixed in this G-action.
   c) (2 pts) Taking a standard basis for M and for $\mathcal{B}$ we may express a bilinear form f by its Gram matrix A_f, and the action of $g \in G$ on M by its matrix $\rho(g)$. Which of the following gives the right action of G on $\mathcal{B}$ (pun intended): (i) $A_f \mapsto \rho(g)^T A_f \rho(g)$, or (ii) $A_f \mapsto \rho(g) A_f \rho(g)^T$?

3. Let $G = C_3 = \langle g \rangle$ be cyclic of order 3 and let $k = \mathbb{F}_3$. We define $M_2 = ke_1 \oplus ke_2$ to be a 2-dimensional space acted on by g via the matrix $\rho(g) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
   a) (1 pts) Find the matrix via which g acts on the space $\mathcal{B}$ of bilinear forms $M \times M \to k$.
   b) (2 pts) Show that the space of G-invariant bilinear forms has dimension 2.
   c) (1 pts) Show that $M_2 \cong M_2^*$ as kG-modules and find a G-invariant non-degenerate form on M_2.
   d) (2 pts) Show that M_2 does not admit any *symmetric* G-invariant non-degenerate bilinear form, but that it does admit a skew-symmetric such form.

4. (1 pts) Let U be a kG-submodule of the kG-module M. Show that U° is a kG-submodule of M^*.
   (3 pts) Suppose further that M comes supplied with a non-singular G-invariant bilinear form. Show that $U^\perp \cong U^\circ$ as kG-modules. Deduce that the isomorphism type of $U^\perp$ is independent of the choice of non-singular G-invariant bilinear form.

5. (2 pts) Let H be a subgroup of a group G, and write

$$H \backslash G = \{ Hg \mid g \in G \}$$

for the set of right cosets of H in G. There is a permutation action of G on this set from the right, namely $(Hg_1)g_2 = Hg_1g_2$. Let $\overline{H} = \sum_{h \in H} h \in kG$ denote the sum of the elements of H, as an element of the group ring of G. Show that the permutation module $k[H\backslash G]$ is isomorphic as an kG-module to the submodule $\overline{H} \cdot kG$ of kG. [Facts about permutation modules for those new to representation theory. These comments will not help with the question in any way that I can see.

a) If Ω is a transitive G-set and $\omega \in \Omega$ with stabilizer $H = \text{Stab}(\omega)$ then $\Omega \cong H\backslash G$ as G-sets.

b) $k[H\backslash G] \cong k \uparrow_H^G$ as kG-modules.]

6. (3=1+2 pts) Let V be the subspace of the 10-dimensional space k^{10} over the field k which has as a basis the vectors

$$\begin{array}{llllllllll}
[0, & 1, & -1, & -1, & 1, & 0, & 0, & 0, & 0, & 0] \\
[1, & 0, & -1, & -1, & 0, & 1, & 0, & 0, & 0, & 0] \\
[0, & 1, & -1, & 0, & 0, & 0, & -1, & 1, & 0, & 0] \\
[1, & 0, & -1, & 0, & 0, & 0, & -1, & 0, & 1, & 0] \\
[1, & 0, & 0, & 0, & -1, & 0, & -1, & 0, & 0, & 1].
\end{array}$$

With respect to this basis of V, write down the Gram matrix for the bilinear form on V which is the restriction of the standard bilinear form on k^{10}. Supposing further that k has characteristic 3, determine the dimension of the space $V/(V \cap V^{\perp})$. [V is the Specht module $S^{[3,2]}$.]

Extra questions for practice with partitions: do not hand in.

7. Find all pairs of partitions of 7 which are not comparable in the dominance ordering, i.e. pairs (λ, μ) for which it is neither true that $\lambda \trianglerighteq \mu$ nor $\mu \trianglerighteq \lambda$.

8. Determine all natural numbers n and partitions λ of n for which the number of λ-tabloids is 12 or fewer (and hence gain an impression of the examples that it is feasible to work by hand).