The second statement is false; since neither $\vec{\mathbf{u}}$ nor $\vec{\mathbf{w}}$ is a multiple of the other, we must have $|\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}| < |\vec{\mathbf{u}}||\vec{\mathbf{w}}|$.

Computations: $|\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}| = 28; \quad |\vec{\mathbf{u}}||\vec{\mathbf{v}}| = \sqrt{14}\sqrt{56} = 28.$

$|\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}| = 20; \quad |\vec{\mathbf{u}}||\vec{\mathbf{w}}| = \sqrt{14}\sqrt{40} = 4\sqrt{35} \approx 23.6$

c. By proposition 1.4.14, $\vec{\mathbf{w}}$ lies clockwise from $\vec{\mathbf{v}}$, since $\det[\vec{\mathbf{v}}\vec{\mathbf{w}}] = v_1w_2 - v_2w_1$ is negative.

d. There is no limit to how long $\vec{\mathbf{w}}$ can be. For example, you can take w_1 to be anything, set $w_2 = 0$, and solve $w_3 = \dfrac{42 - w_1}{3}$. The shortest it can be is $3\sqrt{14}$: by Schwarz's inequality,

$$42 = \vec{\mathbf{v}}\cdot\vec{\mathbf{w}} \le |\vec{\mathbf{v}}||\vec{\mathbf{w}}| = \sqrt{14}|\vec{\mathbf{w}}|, \quad \text{so} \quad |\vec{\mathbf{w}}| \ge \frac{42}{\sqrt{14}} = 3\sqrt{14}.$$

1.4.11 a. True by theorem 1.4.5, because $\vec{\mathbf{w}} = -2\vec{\mathbf{v}}$.

b. False; $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$ is a number; $|\vec{\mathbf{u}}|(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$ is a number times a vector, i.e., a vector.

c. True: since $\vec{\mathbf{w}} = -2\vec{\mathbf{v}}$, we have $\det[\vec{\mathbf{u}},\vec{\mathbf{v}},\vec{\mathbf{w}}] = 0$ and $\det[\vec{\mathbf{u}},\vec{\mathbf{w}},\vec{\mathbf{v}}] = 0$.

d. False, since $\vec{\mathbf{u}}$ is not necessarily (in fact almost surely not) a multiple of $\vec{\mathbf{w}}$; the correct statement is $|\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}| \le |\vec{\mathbf{u}}||\vec{\mathbf{w}}|$.

e. True. f. True.

Solution 1.4.11, part c: Our answer depended on the vectors chosen. In general,

$$\det[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] = -\det[\vec{\mathbf{a}},\vec{\mathbf{c}},\vec{\mathbf{b}}];$$

the result here is true only because both sides are 0, since $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ are linearly dependent.

1.4.12 a. Compute

$$|\vec{\mathbf{v}}| = |\vec{\mathbf{v}}+\vec{\mathbf{w}}-\vec{\mathbf{w}}| \le |\vec{\mathbf{v}}+\vec{\mathbf{w}}| + |-\vec{\mathbf{w}}| = |\vec{\mathbf{v}}+\vec{\mathbf{w}}| + |\vec{\mathbf{w}}|,$$

then subtract $|\vec{\mathbf{w}}|$ from both sides.

b. True: $|\det[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]| = |\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})| = |\vec{\mathbf{a}}^\top(\vec{\mathbf{b}}\times\vec{\mathbf{c}})| \le |\vec{\mathbf{a}}^\top||\vec{\mathbf{b}}\times\vec{\mathbf{c}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}\times\vec{\mathbf{c}}|$. This says that the volume of the parallelepiped spanned by the three vectors (given by $|\det[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]|$) is less than or equal to the length of $\vec{\mathbf{a}}$ times the area of the parallelogram spanned by $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ (that area given by $|\vec{\mathbf{b}}\times\vec{\mathbf{c}}|$).

1.4.13 $\begin{bmatrix} xa \\ xb \\ xc \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} xbc - xbc \\ -(xac - xac) \\ xab - xab \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

1.4.14

a. $\vec{\mathbf{u}}\times(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = \vec{\mathbf{u}}\times\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} \ne (\vec{\mathbf{u}}\times\vec{\mathbf{v}})\times\vec{\mathbf{w}} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}\times\vec{\mathbf{w}} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$

b. $\vec{\mathbf{v}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 0$. The vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{v}}\times\vec{\mathbf{w}}$ are orthogonal.

1.4.15 $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf - ce \\ cd - af \\ ae - bd \end{bmatrix} = -\begin{bmatrix} d \\ e \\ f \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\begin{bmatrix} ec - bf \\ af - cd \\ bd - ae \end{bmatrix}$

1.4.16 a. The area is $\left|\det\begin{bmatrix}1 & 5\\2 & 1\end{bmatrix}\right| = |1-10| = 9.$ b. area $= \left|\det\begin{bmatrix}1 & 5\\2 & -1\end{bmatrix}\right| = 11$

1.4.17 a. It is the line of equation $\begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}2\\-1\end{bmatrix} = 2x-y=0.$

b. It is the line of equation $\begin{bmatrix}x-2\\y-3\end{bmatrix}\cdot\begin{bmatrix}2\\-4\end{bmatrix} = 2x-4-4y+12=0,$ which you can rewrite as $2x-4y+8=0$.

1.4.18 a. The area A of the parallelogram is

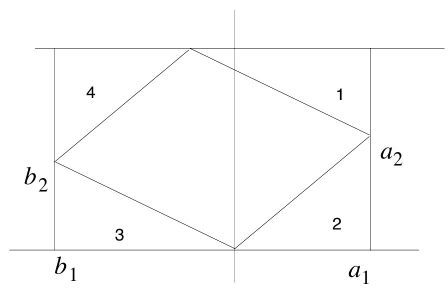
$$(a_1+b_1)(a_2+b_2)-A(1)-A(2)-A(3)-A(4)-A(6),$$

where $A(n)$ is the area of piece n.

$$A(1)=a_2b_1, A(2)=\frac{a_1a_2}{2}, A(3)=\frac{b_1b_2}{2}, A(4)=\frac{b_1b_2}{2}, A(5)=\frac{a_1a_2}{2} \quad\text{and}\quad A(6)=a_2b_1.$$

So $A=(a_1+b_1)(a_2+b_2)-A(1)-A(2)-A(3)-A(4)-A(6)=a_1b_2-a_2b_1$ which is the required result.

b. When b_1 is negative, the area of the rectangle in the figure below is $(a_1-b_1)(a_2+b_2)$, and the area of the parallelogram is that total area minus the area of the triangles marked 1,2,3, and 4.



Since those areas are

$$A(1)=A(3)=\frac{-b_1b_2}{2} \quad\text{and}\quad A(2)=A(4)=\frac{a_1a_2}{2},$$

we have

$$(a_1-b_1)(a_2+b_2)-a_1a_2+b_1b_2=a_1b_2-a_2b_1.$$

1.4.19 a. The length of $\vec{\mathbf{v}}_n$ is $|\vec{\mathbf{v}}_n|=\sqrt{1+\cdots+1}=\sqrt{n}$.

b. The angle is $\arccos\frac{1}{\sqrt{n}}$, which tends to 0 as $n\to\infty$.

Solution 1.4.19, part b: We find it surprising that the diagonal vector $\vec{\mathbf{v}}_n$ is almost orthogonal to all the standard basis vectors when n is large.

1.4.20 This is easy if one remembers that

$$\det A = a_1(b_2c_3-b_3c_2)+b_1(a_3c_2-a_2c_3)+c_1(a_2b_3-a_3b_2)$$

as well as the formulas for developing $\det A$ from the other rows. If we ignore the $\frac{1}{\det A}$ this tells us that along the diagonal of the product of the original matrix and its putative inverse, we find $\det A$, which cancels with the ignored $\frac{1}{\det A}$ to produce 1s. Off the diagonal, the result is the determinant of a 3×3 matrix with two identical rows, which is 0. Therefore the formula is correct.

1.4.21 a. $|A| = \sqrt{1+1+4} = \sqrt{6}$; $|B| = \sqrt{5}$; $|\vec{\mathbf{c}}| = \sqrt{10}$

b. $|AB| = \left|\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}\right| = \sqrt{12} \ \le \ \sqrt{30} = |A||B|$

$|A\vec{\mathbf{c}}| = \sqrt{50} \ \le \ |A||\vec{\mathbf{c}}| = \sqrt{60}$; $|B\vec{\mathbf{c}}| = \sqrt{13} \ \le \ |B||\vec{\mathbf{c}}| = \sqrt{50}$

1.4.22 a. It is an equality when the transpose of each row of the matrix A is the product by a scalar of the vector $\vec{\mathbf{b}}$. In that case, the inequality marked (2) in Equation 1.4.31 is an equality, so the inequality of Equation 1.4.31 becomes an equality.

Solution 1.4.23, part a: This uses the fact that

$$1^2+2^2+\cdots+n^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}.$$

b. It is an equality when all the columns of B are linearly dependent, and the transpose of each row of A is the product by a scalar by any (hence every) column of B. In that case, and in that case only, the inequality in Equation 1.4.33 is an equality by part (a). For example, if $A = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{bmatrix}$, then $|AB| = |A|\,|B|$.

Solution b uses the notion of linear dependence, not introduced until chapter 2. Here is an alternative:

It is an equality if either A or B is the zero matrix; in that case, the other matrix can be anything. Otherwise, it is an equality if and only if all the rows of A and all the columns of B are multiples of some one vector: The first inequality in equation 1.4.33 is an equality if for *each* j all rows of A are multiples of $\vec{\mathbf{b}}_j$. Therefore, *all* columns of B and all rows of A must be multiples of the same vector.

1.4.23 a. The length is $|\vec{\mathbf{w}}_n| = \sqrt{1+4+\cdots+n^2} = \sqrt{\dfrac{n^3}{3}+\dfrac{n^2}{2}+\dfrac{n}{6}}$.

b. The angle $\alpha_{n,k}$ is $\arccos \dfrac{k}{\sqrt{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}}$.

c. In all three cases, the limit is $\pi/2$. Clearly $\lim_{n\to\infty}\alpha_{n,k} = \pi/2$, since the cosine tends to 0.

The limit of $\alpha_{n,n}$ is $\lim_{n\to\infty}\arccos 0 = \pi/2$.

The limit of $\alpha_{n,[n/2]}$ is also $\pi/2$, since it is the arccos of

$$\frac{[n/2]}{\sqrt{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}}, \quad \text{which tends to 0 as } n\to\infty.$$

1.4.24 a. To show that $\vec{\mathbf{v}}^\perp$ is a subspace of $\mathbb{R}^n$ we must show that if $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2 \in \vec{\mathbf{v}}^\perp$, then $(\vec{\mathbf{w}}_1+\vec{\mathbf{w}}_2) \in \vec{\mathbf{v}}^\perp$ and $a\vec{\mathbf{w}}_1 \in \vec{\mathbf{v}}^\perp$ for any $a \in \mathbb{R}$: i.e., that $(\vec{\mathbf{w}}_1 + \vec{\mathbf{w}}_2) \cdot \vec{\mathbf{v}} = 0$ and $a\vec{\mathbf{w}}_1 \cdot \vec{\mathbf{v}} = 0$. The dot product is distributive, so $(\vec{\mathbf{w}}_1 + \vec{\mathbf{w}}_2) \cdot \vec{\mathbf{v}} = \vec{\mathbf{w}}_1 \cdot \vec{\mathbf{v}} + \vec{\mathbf{w}}_2 \cdot \vec{\mathbf{v}} = 0$. Multiplication is distributive, so $\vec{\mathbf{v}} \cdot a\vec{\mathbf{w}}_1 = aw_1v_1 + \cdots + aw_nv_n = a(w_1v_1 + \cdots + w_nv_n) = a\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}_1 = 0$.

b.

$$(\vec{\mathbf{a}} - \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}\vec{\mathbf{v}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} - \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} - \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}|\vec{\mathbf{v}}|^2 = \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} - \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = 0.$$

c. Suppose that $\vec{\mathbf{a}} + t(\vec{\mathbf{a}})\vec{\mathbf{v}} \in \vec{\mathbf{v}}^\perp$. Working this out, we get

$$0 = (\vec{\mathbf{a}} + t(\vec{\mathbf{a}})\vec{\mathbf{v}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} + t(\vec{\mathbf{a}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}),$$

which gives

$$t(\vec{\mathbf{a}}) = -\frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}.$$

This is well-defined since $\vec{\mathbf{v}} \neq \mathbf{0}$, and with this value of $t(\vec{\mathbf{a}})$ we do have $\vec{\mathbf{a}} + t(\vec{\mathbf{a}})\vec{\mathbf{v}} \in \vec{\mathbf{v}}^\perp$ by part (b).

1.4.25

$$\begin{aligned}\left(\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}\right)^2 - \left(x_1y_1 + x_2y_2\right)^2 &= (x_2y_1)^2 + (x_1y_2)^2 - 2x_1x_2y_1y_2 \\ &= (x_1y_2 - x_2y_1)^2 \geq 0,\end{aligned}$$

so $(x_1y_1 + x_2y_2)^2 \leq \left(\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}\right)^2$, so

$$|x_1y_1 + x_2y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

1.4.26 a. The angle is given by

$$\begin{aligned}\alpha\begin{pmatrix} x \\ y \end{pmatrix} &= \arccos \frac{\begin{bmatrix} x \\ y \end{bmatrix} \cdot A \begin{bmatrix} x \\ y \end{bmatrix}}{\left|\begin{bmatrix} x \\ y \end{bmatrix}\right| \left|A \begin{bmatrix} x \\ y \end{bmatrix}\right|} \\ &= \frac{x^2 + xy + 4y^2}{\sqrt{(x^2 + y^2)((x - 2y)^2 + (3x + 4y)^2)}}.\end{aligned}$$

b. This is never 0 when $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Indeed, the numerator can be written

$$x^2 + xy + y^2/4 + 15y^2/4 = (x + y/2)^2 + 15y^2/4,$$

which is the sum of two squares, hence positive unless both are 0. This requires that $y = 0$, hence $x = -y/2 = 0$.

1.4.27 a. To show that the transformation is linear, we need to show that

$$T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}}) + T_{\vec{\mathbf{a}}}(\vec{\mathbf{w}}) \quad \text{and} \quad \alpha T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}}) = T_{\vec{\mathbf{a}}}(\alpha\vec{\mathbf{v}}).$$

For the first,

$$T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}}+\vec{\mathbf{w}}) = \vec{\mathbf{v}}+\vec{\mathbf{w}}-2\big(\vec{\mathbf{a}}\cdot(\vec{\mathbf{v}}+\vec{\mathbf{w}})\big)\vec{\mathbf{a}} = \vec{\mathbf{v}}+\vec{\mathbf{w}}-2(\vec{\mathbf{a}}\cdot\vec{\mathbf{v}}+\vec{\mathbf{a}}\cdot\vec{\mathbf{w}})\vec{\mathbf{a}} = T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}})+T_{\vec{\mathbf{a}}}(\vec{\mathbf{w}}).$$

For the second,

$$\alpha T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}}) = \alpha\vec{\mathbf{v}} - 2\alpha(\vec{\mathbf{a}}\cdot\vec{\mathbf{v}})\vec{\mathbf{a}} = \alpha\vec{\mathbf{v}} - 2(\vec{\mathbf{a}}\cdot\alpha\vec{\mathbf{v}})\vec{\mathbf{a}} = T_{\vec{\mathbf{a}}}(\alpha\vec{\mathbf{v}}).$$

b. We have $T_{\vec{\mathbf{a}}}(\vec{\mathbf{a}}) = -\vec{\mathbf{a}}$, since $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = a^2+b^2+c^2 = 1$:

$$T_{\vec{\mathbf{a}}}(\vec{\mathbf{a}}) = \vec{\mathbf{a}} - 2(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\vec{\mathbf{a}} = \vec{\mathbf{a}} - 2\vec{\mathbf{a}} = -\vec{\mathbf{a}}.$$

If $\vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{a}}$, then $T_{\vec{\mathbf{a}}}(\vec{\mathbf{v}}) = \vec{\mathbf{v}}$, since in that case $\vec{\mathbf{a}}\cdot\vec{\mathbf{v}} = 0$. Thus $T_{\vec{\mathbf{a}}}$ is reflection in the plane that goes through the origin and is perpendicular to $\vec{\mathbf{a}}$.

The transformation $T_{\vec{\mathbf{a}}}$ is the 3-dimensional version of the transformation shown in figure 1.3.4.

c. The matrix of $T_{\vec{\mathbf{a}}}$ is

$$M = [T_{\vec{\mathbf{a}}}(\vec{\mathbf{e}}_1)\,, T_{\vec{\mathbf{a}}}(\vec{\mathbf{e}}_2)\,, T_{\vec{\mathbf{a}}}(\vec{\mathbf{e}}_3)] = \begin{bmatrix} 1-2a^2 & -2ab & -2ac \\ -2ab & 1-2b^2 & -2bc \\ -2ac & -2bc & 1-2c^2 \end{bmatrix}$$

Squaring the matrix gives the 3×3 identity matrix: if you reflect a vector, then reflect it again, you are back to where you started.

1.4.28

1.5.1 a. The set $\{\, x\in\mathbb{R} \mid 0 < x \le 1 \,\}$ is neither open nor closed: the point 1 is in the set, but $1+\epsilon$ is not for every ϵ, showing it isn't open, and 0 is not but $0+\epsilon$ is for every $\epsilon > 0$, showing that the complement is also not open, so the set is not closed.

b. open c. neither d. closed e. closed f. neither g. both.

1.5.2 a. The (x,y)-plane in $\mathbb{R}^3$ is not open; you cannot move in the z direction and stay in the (x,y)-plane. It is closed because its complement is open: any point in $\{\mathbb{R}^3 - (x,y)\text{-plane}\}$ can be surrounded by an open 3-dimensional ball in $\{\mathbb{R}^3 - (x,y)\text{-plane}\}$.

b. The set $\mathbb{R}\subset\mathbb{C}$ is not open: the ball of radius $\epsilon > 0$ around a real number x always contains the non-real number $x+i\epsilon/2$. It is closed because its complement is open; if $z = x+iy \in \{\mathbb{C}-\mathbb{R}\}$, i.e., if $y \ne 0$, then the ball of radius $|y|/2$ around z is contained in $\{\mathbb{C}-\mathbb{R}\}$.

c. The line $x = 5$ in the (x,y)-plane is closed; any point in its complement can be surrounded by an open ball in the complement.

d. The set $(0,1)\subset\mathbb{C}$ is not open since (for example) the point $0.5\in\mathbb{R}$ cannot be surrounded by an open ball in $\mathbb{R}$. It is not closed because its complement is not open. For example, the point $\begin{pmatrix}1\\0\end{pmatrix}\in\mathbb{C}$, cannot be surrounded by an open ball in $\{\mathbb{C}-(0,1)\subset\mathbb{C}\}$.

e. $\mathbb{R}^n\subset\mathbb{R}^n$ is open. It is also closed, because its complement, the empty set, is trivially open.

g. The unit 2-sphere $S \subset \mathbb{R}^3$ is not open: if $\mathbf{x} \in S^2$ and $\epsilon > 0$, then the point $(1+\epsilon/2)\mathbf{x}$ is in $B_\epsilon(\mathbf{x})$ but not on S^2. It is closed, since its complement is open: if $\mathbf{y} \notin S^2$, i.e., if $|\mathbf{y}| \neq 1$, then the open ball $B_{||\mathbf{y}|-1|/2}(\mathbf{y})$ does not intersect S^2.

1.5.3 a. Suppose $A_i, i \in I$ is some collection (probably infinite) of open sets. If $\mathbf{x} \in \bigcup_{i\in I} A_i$, then $\mathbf{x} \in A_j$ for some j, and since A_j is open, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset A_j$. But then $B_\epsilon(\mathbf{x}) \subset \bigcup_{i\in I} A_i$.

Solution 1.5.3, part b: There is a smallest ϵ_i, because there are finitely many of them, and it is positive. If there were infinitely many, then there would be a greatest lower bound, but it could be 0.

Part c: In fact, every closed set is a countable intersection of open sets.

b. If $A_1, \dots, A_j$ are open and $\mathbf{x} \in \cap_{i=1}^k A_i$, then there exist $\epsilon_1, \dots, \epsilon_k > 0$ such that $B_{\epsilon_i}(\mathbf{x}) \subset A_i$, for $i = 1, \dots, k$. Set ϵ to be the smallest of $\epsilon_1, \dots, \epsilon_k$. Then $B_\epsilon(\mathbf{x}) \subset B_{\epsilon_i}(\mathbf{x}) \subset A_i$.

c. The infinite intersection of open sets $(-1/n, 1/n)$, for $n = 1, 2, \dots$, is not open; as $n \to \infty$, $-1/\infty \to 0$ and $1/\infty \to 0$; the set $\{0\}$ is not open.

1.5.4

1.5.5 a. This set is open. Indeed, if you choose $\begin{pmatrix} x \\ y \end{pmatrix}$ in your set, then $1 < \sqrt{x^2+y^2} < \sqrt{2}$. Set

$$r = \min\left\{\sqrt{x^2+y^2} - 1, \sqrt{2} - \sqrt{x^2+y^2}\right\} > 0.$$

Then the ball of radius r around $\begin{pmatrix} x \\ y \end{pmatrix}$ is contained in the set, since if $\begin{pmatrix} u \\ v \end{pmatrix}$ is in that ball, then, by the triangle inequality,

The first equation uses the familiar form of the triangle inequality: if $\mathbf{a} = \mathbf{b} + \mathbf{c}$, then

$$|\mathbf{a}| \leq |\mathbf{b}| + |\mathbf{c}|.$$

The second uses the variant

$$|\mathbf{a}| \geq \Big||\mathbf{b}| - |\mathbf{c}|\Big|.$$

$$\left|\begin{bmatrix} u \\ v \end{bmatrix}\right| \leq \left|\begin{bmatrix} u-x \\ v-y \end{bmatrix}\right| + \left|\begin{bmatrix} x \\ y \end{bmatrix}\right| < r + \left|\begin{bmatrix} x \\ y \end{bmatrix}\right| \leq \sqrt{2}$$

$$\left|\begin{bmatrix} u \\ v \end{bmatrix}\right| \geq \left|\begin{bmatrix} x \\ y \end{bmatrix}\right| - \left|\begin{bmatrix} u-x \\ v-y \end{bmatrix}\right| > \left|\begin{bmatrix} x \\ y \end{bmatrix}\right| - r \geq 1.$$

b. The locus $xy \neq 0$ is also open. It is the complement of the two axes, so that if $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the set, then $r = \min\{|x|, |y|\} > 0$, and the ball B of radius r around $\begin{pmatrix} x \\ y \end{pmatrix}$ is contained in the set. Indeed, if $\begin{pmatrix} u \\ v \end{pmatrix}$ is B, then $|u| = |x + u - x| > |x| - |u - x| > |x| - r \geq 0$, so u is not 0, and neither is v, by the same argument.

Many students have found exercise 1.5.5 difficult, even though they also thought it was obvious, but didn't know how to say it. If this applies to you, you should check carefully where we used the triangle inequality, and how. Almost everything concerning inequalities requires the triangle inequality.

c. This time our set is the x-axis, and it is closed. We will use the criterion that a set is closed if the limit of a convergent sequence of elements of the set is in the set (proposition 1.5.17). If $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ is a sequence in the set, and converges to $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, then all $y_n = 0$, so $y_0 = \lim_{n\to\infty} y_n = 0$, and the limit is also in the set.

d. The rational numbers are neither open nor closed. Any rational number x is the limit of the numbers $x + \sqrt{2}/n$, which are all irrational, so the rationals aren't closed. Any irrational number is the limit of the finite decimals used to write it, which are all rational, so the irrationals aren't closed either.