1.5.6

1.5.7 a. The natural domain is $\mathbb{R}^2$ minus the union of the two axes; it is open.

b. The natural domain is that part of $\mathbb{R}^2$ where $x^2 > y$ (i.e., the area "inside" the parabola of equation $y = x^2$). It is open since its "fence" x^2 belongs to its neighbor.

c. The natural domain of $\ln \ln x$ is $\{x | x > 1\}$, since we must have $\ln x > 0$. This domain is open.

d. The natural domain of arcsin is $[-1, 1]$. Thus the natural domain of $\arcsin \frac{3}{x^2+y^2}$ is $\mathbb{R}^2$ minus the open disc $x^2 + y^2 < 3$. Since this domain is the complement of an open disc it is closed and not open.

e. The natural domain is all of $\mathbb{R}^2$, which is open.

f. The natural domain is $\mathbb{R}^3$ minus the union of the three coordinate planes of equation $x = 0$, $y = 0$, $z = 0$; it is open.

1.5.8 a. The matrix A is

$$A = \begin{bmatrix} 0 & -\epsilon & -\epsilon \\ 0 & 0 & -\epsilon \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{since} \quad \underbrace{\begin{bmatrix} 1 & \epsilon & \epsilon \\ 0 & 1 & \epsilon \\ 0 & 0 & 1 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} - \underbrace{\begin{bmatrix} 0 & -\epsilon & -\epsilon \\ 0 & 0 & -\epsilon \\ 0 & 0 & 0 \end{bmatrix}}_{A}.$$

To compute the inverse of B, i.e., B^{-1}, we compute the series based on A:

$$B^{-1} = (I - A)^{-1} = I + A + A^2 + A^3 \dots .$$

We have

$$A^2 = \begin{bmatrix} 0 & 0 & \epsilon^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\epsilon & -\epsilon \\ 0 & 0 & -\epsilon \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \epsilon^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon & -\epsilon + \epsilon^2 \\ 0 & 1 & -\epsilon \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case ϵ doesn't need to be small for the series to converge.

b.

$$\underbrace{\begin{bmatrix} 1 & -\epsilon \\ +\epsilon & 1 \end{bmatrix}}_{C} = I - \underbrace{\begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix}}_{A}.$$

To compute $C^{-1} = (I - A)^{-1} = I + A + A^2 + A^3 \dots$, first compute A^2, A^3, and A^4:

$$\begin{array}{cccc} & \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix} & \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix} & \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix} & \underbrace{\begin{bmatrix} -(\epsilon^2) & 0 \\ 0 & -(\epsilon^2) \end{bmatrix}}_{A^2} & \underbrace{\begin{bmatrix} 0 & -(\epsilon^3) \\ \epsilon^3 & 0 \end{bmatrix}}_{A^3} & \underbrace{\begin{bmatrix} \epsilon^4 & 0 \\ 0 & \epsilon^4 \end{bmatrix}}_{A^4}. \end{array}$$

In the series $I+A+A^2+A^3$, each entry of A itself converges to a limit: $S = a + ar + ar^2 + \cdots = \frac{a}{a-r}$. For $a_{1,1}$, we have $a = 1, r = -\epsilon^2$, so $a_{1,1}$ converges to $\frac{1}{1+\epsilon^2}$. For $a_{1,2}$, we have $a = \epsilon, r = -\epsilon^2$, so $a_{1,2}$ converges to $\frac{\epsilon}{1+\epsilon^2}$, and so on. In this way we get

Note that according to proposition 1.5.37, we need $|A| < 1$ for our series $I + A + A^2 \dots$ to be convergent. Since $|A| = \epsilon\sqrt{2}$, this would mean we need to have $|\epsilon| < 1/\sqrt{2}$. But in fact all we need in this case is $\epsilon < 1$.

$$C^{-1} = \begin{bmatrix} \frac{1}{1+\epsilon^2} & \frac{\epsilon}{1+\epsilon^2} \\ \frac{-\epsilon}{1+\epsilon^2} & \frac{1}{1+\epsilon^2} \end{bmatrix}.$$

1.5.9 For any $n > 0$ we have $\left|\sum_{i=1}^{n} \mathbf{x}_i\right| \le \sum_{i=1}^{n} |\mathbf{x}_i|$ by the triangle inequality (theorem 1.4.9). Because $\sum_{i=1}^{\infty} \mathbf{x}_i$ converges, $\sum_{i=1}^{n} \mathbf{x}_i$ converges as $n \to \infty$. So :

$$\left|\sum_{i=1}^{\infty} \mathbf{x}_i\right| \le \sum_{i=1}^{\infty} |\mathbf{x}_i|.$$

1.5.10 a. More generally, if $f(x) = a_0 + a_1 x + \dots$ is any power series which converges for $|x| < R$, the series of square $n \times n$ matrices

$$f(A) = a_0 + a_1 A + a_2 A^2 + \dots$$

converges for $|A| < R$. Indeed, for $|x| < R$ the power series

$$\sum_{i=0}^{\infty} |a_i|\ |x|^i$$

converges absolutely, so the series of matrices also converges absolutely by propositions 1.5.34 and 1.4.11:

$$\sum_{k=1}^{\infty} |a_k A^k| \le \sum_{k=1}^{\infty} |a_k||A|^k.$$

In particular, the exponential series defining e^A converges for all A. Finding an actual bound is a little irritating because the length of the $n \times n$ identity matrix is not 1; it is $\sqrt{n}$. We deal with this by adding and subtracting 1:

$$|e^A| = \left|I + A + \frac{A^2}{2!} + \cdots\right| \le |I| + |A| + \left|\frac{A^2}{2!}\right| + \cdots \le \sqrt{n} + |A| + \frac{|A|^2}{2!} + \cdots$$
$$= \sqrt{n} - 1 + \left(1 + |A| + \frac{|A|^2}{2!} + \cdots\right) = \sqrt{n} - 1 + e^{|A|}.$$

When we start using norms of matrices rather than lengths (section 2.9) this nastiness of the length of the identity matrix disappears.

b.

$$(1) \qquad e^{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \tfrac{1}{2!}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}.$$

(2) Note the remarkable fact that $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so

$$e^{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

(3) Let us compute a few powers:

$$\begin{array}{cccccc} & \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} & \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} & \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} & \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} & \dots \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} & \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix} & \begin{bmatrix} 0 & -a^3 \\ a^3 & 0 \end{bmatrix} & \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix} & \dots \\ I & A & A^2 & A^3 & A^4 & \dots \end{array}$$

We see that only the even terms contribute to the diagonal, and only the odd terms contribute to the antidiagonal. We can rewrite the series as

$$e^{\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}} = \begin{bmatrix} 1 - a^2/2 + a^4/4! + \dots & a - a^3/3! + a^5/5! + \dots \\ -a + a^3/3! - a^5/5! + \dots & 1 - a^2/2 + a^4/4! + \dots \end{bmatrix} = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix},$$

where we have recognized what we hope are old friends, the power series for $\sin x$ and $\cos x$, in the diagonal and antidiagonal terms respectively.

c. (1) If we set $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, we have

$$e^{A+B} = \begin{bmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{bmatrix} \quad \text{but} \quad e^A e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrices A and B above do not commute:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

When $AB = BA$, the formula $e^{A+B} = e^A e^B$ is true. Indeed,

$$\begin{aligned} e^{A+B} &= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots \\ &= I + (A+B) + \frac{1}{2!}(A^2 + AB + BA + B^2) \\ &\quad + \frac{1}{3!}(A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3) + \cdots \end{aligned}$$

whereas

$$\begin{aligned} e^A e^B &= \left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots\right)\left(I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \dots\right) \\ &= I + (A+B) + \frac{1}{2!}(A^2 + 2AB + B^2) + \frac{1}{3!}(A^3 + 3A^2B + 3AB^2 + B^3) + \dots \end{aligned}$$

We see that the series are equal when $AB = BA$, and have no reason to be equal otherwise. This proof is just a little shaky; the terms of the series don't quite come in the same order, and we need to invoke the fact that for absolutely convergent series, we can rearrange the terms in any order, and the series still converges to the same sum.

(2) It follows from the above that $e^{2A} = \left(e^A\right)^2$: of course, A commutes with itself.

1.5.11 a. First let us see that

$$\Big((\forall \epsilon > 0)(\exists N)(n > N) \implies |\mathbf{a}_n - \mathbf{a}| < \varphi(\epsilon)\Big) \implies \big(\mathbf{a}_n \text{ converges to } \mathbf{a}\big).$$

Choose $\eta > 0$. Since $\lim_{t\to 0}\varphi(t) = 0$, there exists $\delta > 0$ such that when $0 < t \le \delta$ we have $\varphi(t) < \eta$. Our hypothesis guarantees that there exists N such that when $n > N$, then $|\mathbf{a}_n - \mathbf{a}| \le \varphi(\delta) = \eta$.

Now for the converse:

$$\big(\mathbf{a}_n \text{ converges to } \mathbf{a}\big) \implies \Big((\forall \epsilon > 0)(\exists N)(n > N) \implies |\mathbf{a}_n - \mathbf{a}| < \varphi(\epsilon)\Big).$$

For any $\epsilon > 0$, we also have $\varphi(\epsilon) > 0$, so there exists N such that

$$n > N \implies |\mathbf{a}_n - \mathbf{a}| < \varphi(\epsilon).$$

b. The analogous statement for limits of functions is:

Let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that $\lim_{t\to 0}\varphi(t) = 0$. Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}^m$, and $\mathbf{x}_0 \in \overline{U}$. Then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{a}$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that when $\mathbf{x} \in U$ and $|\mathbf{x} - \mathbf{x}_0| < \delta$, we have $|f(\mathbf{x}) - \mathbf{a}| < \varphi(\epsilon)$.

1.5.12 Let us first show the interesting case: (2) $\implies$ (1).

Choose $\epsilon > 0$, and then choose $\gamma > 0$ such that when $|t| \le \gamma$ we have $u(t) < \epsilon$; such a γ exists because $\lim_{t\to 0} u(t) = 0$. Our hypothesis is that for all $\epsilon > 0$ there exists $\delta > 0$ such that when $|\mathbf{x} - \mathbf{x}_0| < \delta$, and $\mathbf{x} \in U$, then $|f(\mathbf{x}) - a| < u(\epsilon)$. Similarly, for all $\gamma > 0$ there exists $\delta > 0$ such that when $|\mathbf{x} - \mathbf{x}_0| < \delta$, and $\mathbf{x} \in U$, then $|f(\mathbf{x}) - a| < u(\gamma)$. Since $u(\gamma) < \epsilon$, this implies $|f(\mathbf{x}) - a| < u(\epsilon)$.

For the other direction, just take $u(\epsilon = \epsilon$.

1.5.13 If every convergent sequence in C converges to a point in C then $\forall$ points a such that $\forall n$ $\exists$ a point $b_n \in C$ such that $|b_n - a| \le \frac{1}{n}$, the sequence b_n converges to a so $a \in C$. It then follows that $\mathbb{R} - C$ is open (a is not in $\mathbb{R} - C$) so C is closed.

1.5.14 a. The functions x and y both are continuous on $\mathbb{R}^2$, so they have limits at all points. Hence so does $x + y$ (the sum of two continuous functions is continuous), and x^2 (the product of two continuous functions is continuous). The quotient of two continuous functions is continuous wherever the denominator is not 0, and $x + y = 3$ at $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So the limit exists, and is 1/3.

b. This is much nastier: the denominator does vanish at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If we let $x = y = t \ne 0$, the function becomes

$$\frac{t\sqrt{|t|}}{2t^2} = \frac{1}{2\sqrt{|t|}}.$$

Evidently this can be made arbitrarily large by taking $|t|$ sufficiently small. Strictly speaking, this shows that the limit does not exist, but sometimes one allows infinite limits. Is the limit ∞? No, because $f\begin{pmatrix} t \\ 0 \end{pmatrix} = 0$, so there also are points arbitrarily close to the origin where the function is zero. So there is no value, even ∞, which the function is close to when $|\begin{pmatrix} x \\ y \end{pmatrix}|$ is small (i.e., the distance from $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\left|\begin{bmatrix} x-0 \\ y-0 \end{bmatrix}\right|$, is small).

c. This time, if we approach the origin along the diagonal, we get

$$f\begin{pmatrix} t \\ t \end{pmatrix} = \frac{|t|}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}},$$

whereas if we approach the origin along the axes, the function is zero, and the limit is zero. Thus the limit does not exist.

d. This is no problem: x^2 is continuous everywhere, y^3 is continuous everywhere, -3 is continuous everywhere, the sum is continuous everywhere, and the limit exists, and is 6.

Solution 1.5.15: Here, x plays the role of the alligators in section 0.2, and $x \geq 0$ satisfying $|-2-x| < \delta$ plays the role of eleven-legged alligators; the conclusion $|\sqrt{x}-5| < \epsilon$ (i.e., that 5 is the limit) is the conclusion "are orange with blue spots" and the conclusion $|\sqrt{x}-3| < \epsilon$ (i.e., that 3 is the limit) is the conclusion "are black with white stripes."

1.5.15 Both statements are true. To show that the first is true, we say: for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying $x \geq 0$ and $|-2-x| < \delta$, then $|\sqrt{x}-5| < \epsilon$. For any $\epsilon > 0$, choose $\delta = 1$. Then there is no $x \geq 0$ satisfying $|-2-x| < \delta$. So for those nonexistent x satisfying $|-2-x| < 1$, it is true that $|\sqrt{x}-5| < \epsilon$. By the same argument the second statement is true.

1.5.16 a. For any $\epsilon > 0$, there exists $\delta > 0$ such that when $0 < \sqrt{x^2+y^2} < \delta$, we have

$$\left| f\begin{pmatrix} x \\ y \end{pmatrix} - a \right| < \epsilon.$$

b. The limit of f does not exist. Indeed, if we set $y = 0$ the limit becomes

$$\lim_{x \to 0} \frac{\sin x}{|x|}.$$

This approaches $+1$ as x tends to 0 through positive values, and tends to -1 as x tends to 0 through negative values.

The limit of g does exist, and is 0. By l'Hôpital's rule (or because you remember it), we have

$$\lim_{x \downarrow 0} x \ln x = \lim_{x \downarrow 0} \frac{\ln x}{1/x} = \lim_{x \downarrow 0} \frac{1/x}{-1/x^2} = \lim_{x \downarrow 0} -x = 0.$$

(We write $x \downarrow 0$ rather than $x \to 0$ to indicate that x is decreasing to 0; $\ln(x)$ is not defined for negative values of x.) When $(x^2+y^4) < 1$ so that the logarithm is negative, we have

$$2|x| \ln |x| + 4|y| \ln |y| = (|x|+|y|) \ln(x^2+y^4) < 0.$$

Given $\epsilon > 0$, find $\delta > 0$ so that when $0 < |x| < \delta$, we have $4|x| \ln |x| > -\frac{2\epsilon}{3}$. When $\sqrt{x^2+y^2} < \delta$, then in particular $|x| < \delta$ and $|y| < \delta$, so that

$$2|x| \ln |x| + 4|y| \ln |y| > -\frac{\epsilon}{3} - \frac{2\epsilon}{3} = -\epsilon, \quad \text{so that} \quad \left|(|x|+|y|) \ln(x^2+y^4)\right| < \epsilon.$$

1.5.17 Set $\mathbf{a} = \lim_{m\to\infty} \mathbf{a}_m,\ \mathbf{b} = \lim_{m\to\infty} \mathbf{b}_m$ and $c = \lim_{m\to\infty} c_m$.

1. Choose $\epsilon > 0$ and find M_1 and M_2 such that if $m \geq M_1$ then we have $|\mathbf{a}_m - \mathbf{a}| \leq \epsilon/2$, and if $m \geq M_2$ then $|\mathbf{b}_m - \mathbf{b}| \leq \epsilon/2$. If

$$m \geq M = \max(M_1, M_2),$$

we have

$$|\mathbf{a}_m + \mathbf{b}_m - \mathbf{a} - \mathbf{b}| \leq |\mathbf{a}_m - \mathbf{a}| + |\mathbf{b}_m - \mathbf{b}| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So the sequence $(\mathbf{a}_m + \mathbf{b}_m)$ converges to $\mathbf{a} + \mathbf{b}$.

2. Choose $\epsilon > 0$. Find M_1 such that if

$$m \geq M_1 \quad \text{then} \quad |\mathbf{a}_m - \mathbf{a}| \leq \frac{1}{2} \inf\left(\frac{\epsilon}{|c|}, \epsilon\right).$$

The inf is there to guard against the possibility that $|c| = 0$. In particular, if $m \geq M_1$, then $|\mathbf{a}_m| \leq |\mathbf{a}| + \epsilon$. Next find M_2 such that if

$$m \geq M_2 \quad \text{then} \quad |c_m - c| \leq \frac{\epsilon}{2(|\mathbf{a}| + \epsilon)}.$$

If $m \geq M = \max(M_1, M_2)$, then

$$\begin{aligned} |c_m\mathbf{a}_m - c\mathbf{a}| &= |c(\mathbf{a}_m - \mathbf{a}) + (c_m - c)\mathbf{a}_m| \\ &\leq |c(\mathbf{a}_m - \mathbf{a})| + |(c_m - c)\mathbf{a}_m| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

so the sequence $(c_m\mathbf{a}_m)$ converges and the limit is $c\mathbf{a}$.

3. We can either repeat the argument above, or use parts 1 and 2 as follows:

$$\begin{aligned} \lim_{m\to\infty} \mathbf{a}_m \cdot \mathbf{b}_m &= \lim_{m\to\infty} \sum_{i=1}^n a_{m,i} b_{m,i} = \sum_{i=1}^n \lim_{m\to\infty} (a_{m,i} b_{m,i}) \\ &= \sum_{i=1}^n \left(\lim_{m\to\infty} a_{m,i}\right)\left(\lim_{m\to\infty} b_{m,i}\right) = \sum_{i=1}^n a_i b_i = \mathbf{a} \cdot \mathbf{b}. \end{aligned}$$

4. Find C such that $|\mathbf{a}_m| \leq C$ for all m; saying that $\mathbf{a}_m$ is bounded means exactly that such a C exists. Choose $\epsilon > 0$, and find M such that when $m > M$, then $|c_m| < \epsilon/C$ (this is possible since the c_m converge to 0). Then when $m > M$ we have

$$|c_m\mathbf{a}_m| = |c_m||\mathbf{a}_m| \leq \frac{\epsilon}{C} C = \epsilon.$$

1.5.18 If c_m is a subsequence of a_n then $\forall n\ \exists m_n$ such that if $m \geq m_n$ then $\exists n_m \geq n$ such that $c_m = a_{n_m}$, so if the sequence a_k converges to a then so does any subsequence (instead of $m \geq n$ we have $m \geq m_n$).

1.5.19 a. Suppose $I - A$ is invertible, and write

$$I - A + C = I - A + C(I - A)^{-1}(I - A) = \big(I + C(I - A)^{-1}\big)(I - A),$$

so

$$\begin{aligned}(I - A + C)^{-1} &= (I - A)^{-1}\Big(I + C(I - A)^{-1}\Big)^{-1} \\ &= (I - A)^{-1}\Bigg(\underbrace{I - (C(I - A)^{-1}) + (C(I - A)^{-1})^2 - (C(I - A)^{-1})^3 + \cdots}_{\text{geometric series}}\Bigg)\end{aligned}$$

so long as the series is convergent. By proposition 1.5.37, this will happen if

$$|C(I - A)^{-1}| < 1, \quad \text{in particular if} \quad |C| < \frac{1}{|(I - A)^{-1}|}.$$

Thus every point of U is the center of a ball contained in U.

For the second part of the question, the matrices

$$C_n = \begin{bmatrix} 1 - 1/n & 0 \\ 0 & 1 - 1/n \end{bmatrix}, \quad n = 1, 2, \dots$$

converge to I, and C_n is in U since $I - C_n = \begin{bmatrix} 1/n & 0 \\ 0 & 1/n \end{bmatrix}$ is invertible.

b. Simply factor: $(A + I)(A - I) = A^2 + A - A - I = A^2 - I$, so

$$(A^2 - I)(A - I)^{-1} = (A + I)(A - I)(A - I)^{-1} = A + I,$$

which converges to $2I$ as $A \to I$.

c. Showing that V is open is very much like showing that U is open (part a). Suppose $B - A$ is invertible, and write

$$B - A + C = (I + C(B - A)^{-1})(B - A),$$

so

$$\begin{aligned}(B - A + C)^{-1} &= (B - A)^{-1}\big(I + C(B - A)^{-1}\big)^{-1} \\ &= (B - A)^{-1}\Big(I - (C(B - A)^{-1}) + (C(B - A)^{-1})^2 - (C(B - A)^{-1})^3 + \cdots\Big)\end{aligned}$$

so long as the series is convergent. This will happen if

$$|C(B - A)^{-1}| < 1, \quad \text{in particular, if} \quad |C| < \frac{1}{|(B - A)^{-1}|}.$$

Thus every point of V is the center of a ball contained in V. Again, the matrices

$$\begin{bmatrix} 1 + 1/n & 0 \\ 0 & -1 + 1/n \end{bmatrix}, \quad n = 1, 2, \dots$$

do the trick.

d. This time, the limit does not exist. Note that you cannot factor $A^2 - B^2 = (A + B)(A - B)$ if A and B do not commute.

First set

$$A_n = \begin{bmatrix} 1/n + 1 & 1/n \\ 0 & -1 + 1/n \end{bmatrix}.$$

Then

$$A_n^2 - B^2 = \begin{bmatrix} 2/n + 1/n^2 & 2/n^2 \\ 0 & -2/n + 1/n^2 \end{bmatrix} \quad \text{and} \quad (A-B)^{-1} = \begin{bmatrix} n & -n \\ 0 & n \end{bmatrix}.$$

Thus we find

$$(A_n^2 - B^2)(A_n - B)^{-1} = \begin{bmatrix} 2+1/n & -2+1/n \\ 0 & -2+1/n \end{bmatrix} \to \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$$

as $n \to \infty$.

Do the same computation with $A'_n = \begin{bmatrix} 1/n+1 & 0 \\ 0 & -1+1/n \end{bmatrix}$. This time we find

$$({A'}_n^2 - B^2)({A'}_n - B)^{-1} = \begin{bmatrix} 2+1/n & 0 \\ 0 & -2+1/n \end{bmatrix} \to \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = 2B$$

as $n \to \infty$.

Since both sequence A_n and A'_n converge to B, this shows that there is no limit.

Part d: You may wonder how we came by the matrices A_n; we observed that

$$B\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

so these matrices do not commute.

1.5.20 a. The powers of A are

$$A^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}, \; A^3 = \begin{bmatrix} 4a^3 & 4a^3 \\ 4a^3 & 4a^3 \end{bmatrix}, \; \dots, \; A^n = \begin{bmatrix} 2^{n-1}a^n & 2^{n-1}a^n \\ 2^{n-1}a^n & 2^{n-1}a^n \end{bmatrix}.$$

For this sequence of matrices to converge to the zero matrix, each entry must converge to 0. This will happen if $|a| < 1/2$ (see Example 0.5.6). The sequence will also converge if $a = 1/2$; in that case the sequence is constant.

b. Exactly as above,

$$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}^n = \begin{bmatrix} 3^{n-1}a^n & 3^{n-1}a^n & 3^{n-1}a^n \\ 3^{n-1}a^n & 3^{n-1}a^n & 3^{n-1}a^n \\ 3^{n-1}a^n & 3^{n-1}a^n & 3^{n-1}a^n \end{bmatrix},$$

so the sequence converges to the 0 matrix if $|a| < 1/3$; it converges when $a = 1/3$ because it is a constant sequence. For an $m \times m$ matrix filled with a's, the same computation shows that A^n will converge to 0 if $|a| < 1/m$. It will converge when $a = 1/m$ because it is a constant sequence.

1.5.21 a. This is a quotient of continuous functions where the denominator does not vanish at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so it is continuous at the origin.

b. Again, there is no problem: this is the square root of a continuous function, at a point where the function is 1, so it is continuous at the origin.

c. If we approach the origin along the x-axis, $f = 1$, and if we approach the origin along the y-axis, $f = |y|^{\frac{2}{3}}$ goes to 0, so f is not continuous at the origin. There is no way of choosing a value of $f\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ that will make f continuous at the origin.

d. When $0 < x^2 + 2y^2 < 1$,

$$\begin{aligned} 0 > (x^2+y^2)\ln(x^2+2y^2) &\geq (x^2+y^2)\ln\big(2(x^2+y^2)\big) \\ &= (x^2+y^2)\ln(x^2+y^2) + (x^2+y^2)\ln 2. \end{aligned}$$

The term $(x^2+y^2)\ln(x^2+y^2)$ tends to 0 using equation (1) in the margin. The second term, $(x^2+y^2)\ln 2$, obviously tends to 0. So if we choose $f\begin{pmatrix}0\\0\end{pmatrix}=0$, then f is continuous.

Part d uses the following statement from one-variable calculus:

$$\lim_{u\to 0} u\ln|u| = 0. \qquad (1)$$

This can be proved by applying l'Hôpital's rule to $\dfrac{\ln|u|}{1/u}$.

e. The function is not continuous near the origin. Since $\ln 0$ is undefined, the diagonal $x+y=0$ is not part of the function's domain of definition. However, the function is defined at points arbitrarily close to that line, e.g., the point $\begin{pmatrix}x\\-x+e^{-1/x^3}\end{pmatrix}$. At this point we have

$$\left(x^2+\left(-x+e^{-1/x^3}\right)^2\right)\ln\left|x-x+e^{-1/x^3}\right| \ge x^2\left|\frac{1}{x^3}\right| = \frac{1}{|x|},$$

which tends to infinity as x tends to 0. But if we approach the origin along the x-axis (for instance), the function is $x^2\ln|x|$, which tends to 0 as x tends to 0.

1.5.22 a. Since $|A|=3$, $\delta=\epsilon/4$ works, by the proof of Theorem 1.5.32.

b. The largest δ can be is $\epsilon/\sqrt{5}$. Indeed, let $\begin{bmatrix}x\\y\end{bmatrix}=r\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}$. Then

$$\left|A\begin{bmatrix}x\\y\end{bmatrix}\right| = r\sqrt{4\cos^2\theta+4\sin^2\theta+\sin^2\theta} = r\sqrt{4+\sin^2\theta}\le\sqrt{5}r,$$

with equality realized when $\theta=\pi/2$, i.e., when $x=0$. If follows that if

$$\left|\begin{bmatrix}x_1\\y_1\end{bmatrix}-\begin{bmatrix}x_2\\y_2\end{bmatrix}\right|\le\delta=\frac{\epsilon}{\sqrt{5}},$$

then

$$\left|A\begin{bmatrix}x_1\\y_1\end{bmatrix}-A\begin{bmatrix}x_2\\y_2\end{bmatrix}\right| = \left|A\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}-\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)\right|\le\sqrt{5}\frac{\epsilon}{\sqrt{5}}=\epsilon.$$

Thus $\delta=\frac{\epsilon}{\sqrt{5}}$ works, and since the inequality above is an equality when $x_1=x_2$, it is the largest δ that works.

1.5.23 a. To say that $\lim_{B\to A}(A-B)^{-1}(A^2-B^2)$ exists means that there is a matrix C such that for all $\epsilon>0$, there exists $\delta>0$, such that when $|B-A|<\delta$ and $B-A$ is invertible, then

$$|(B-A)^{-1}(B^2-A^2)-C|<\epsilon.$$

b. We will show that the limit exists, and is $\begin{bmatrix}2&0\\0&2\end{bmatrix}=2I$. Write $B=I+H$, with H invertible, and choose $\epsilon>0$. We need to show that there exists $\delta>0$ such that if $|H|<\delta$, then

$$\left|(I+H-I)^{-1}(I+H)^2-I^2-2I\right|<\epsilon. \qquad (2)$$

Indeed,

$$\begin{aligned}\left|(I+H-I)^{-1}(I+H)^2-I^2-2I\right| &= \left|H^{-1}(I^2+IH+HI+H^2-I^2)-2I\right|\\ &= \left|H^{-1}(2H+H^2)-2I\right| = |H|.\end{aligned}$$