Solutions for Chapter 2

2.1.1 a. $\begin{bmatrix}3 & 1 & -4\\0 & 2 & 1\\1 & -3 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}\begin{bmatrix}0\\4\\1\end{bmatrix}$ b. $\begin{bmatrix}3 & 1 & -4 & 0\\0 & 2 & 1 & 4\\1 & -3 & 0 & 1\end{bmatrix}$ c. $\begin{bmatrix}1 & -7 & 2 & 1\\1 & -3 & 0 & 2\\2 & -2 & 0 & -1\end{bmatrix}$

2.1.2 a. $\begin{bmatrix}0 & 3 & -1 & 0\\-2 & 1 & 2 & 0\\1 & 0 & -5 & 0\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$.

b. $\begin{bmatrix}2 & 3 & -1 & 1\\0 & -2 & 1 & 2\\1 & 0 & -2 & -1\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & 0 & 5/3\\0 & 1 & 0 & -1/3\\0 & 0 & 1 & 4/3\end{bmatrix}$

2.1.3

a. $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\to\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\end{bmatrix}$ b. $\begin{bmatrix}1 & -1 & 1\\-1 & 0 & 2\\-1 & 1 & 1\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

c. $\begin{bmatrix}1 & 2 & 3 & 5\\2 & 3 & 0 & -1\\0 & 1 & 2 & 3\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 2\end{bmatrix}$

d. $\begin{bmatrix}1 & 3 & -1 & 4\\1 & 2 & 1 & 2\\3 & 7 & 1 & 9\end{bmatrix}\to\begin{bmatrix}1 & 0 & 5 & -2\\0 & 1 & -2 & 2\\0 & 0 & 0 & 1\end{bmatrix}$

e. $\begin{bmatrix}1 & 1 & 1 & 1\\2 & -3 & 3 & 3\\1 & -4 & 2 & 2\end{bmatrix}\to\begin{bmatrix}1 & 0 & 6/5 & 6/5\\0 & 1 & -1/5 & -1/5\\0 & 0 & 0 & 0\end{bmatrix}$

2.1.4 Consider the following sequence of four row operations:

- Add row i to row j;
- Subtract row j from row i;
- Add row i to row j;
- Multiply row i by -1.

If we denote row i by r_i and row j by r_j, this leads to

$$\begin{matrix}r_i\\r_j\end{matrix}\mapsto\begin{matrix}r_i\\r_j+r_i\end{matrix}\mapsto\begin{matrix}-r_j\\r_i+r_j\end{matrix}\mapsto\begin{matrix}-r_j\\r_i\end{matrix}\mapsto\begin{matrix}r_j\\r_i\end{matrix}.$$

Clearly we have exchanged row i and row j.

2.1.5 You can undo "multiplying row i by $m \neq 0$" by "multiplying row i by $1/m$" (which is possible because $m \neq 0$; see definition 2.1.1).

You can undo "adding row i to row j" by "subtracting row i from row j," i.e., "adding ($-$row i) to row j".

You can undo "switching row i and row j" by "switching row i and row j" again.

2.1.6 a. The original matrix corresponds to the set of equations

$$\begin{aligned} 2x + y + 3z &= 1 \\ x - y &= 1 \\ 2x + z &= 1, \end{aligned}$$

with solutions $x = 1/3, y = -2/3, z = 1/3$ since the matrix row reduces to $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$.

The various row operations correspond to the systems of equations

$$\text{(i)} \quad \begin{aligned} 2x + y + 3z &= 1 \\ 2x - 2y &= 2 \\ 2x + z &= 1. \end{aligned} \qquad \text{(ii)} \quad \begin{aligned} x - y &= 1 \\ 2x + y + 3z &= 1 \\ 2x + z &= 1, \end{aligned} \qquad \text{(iii)} \quad \begin{aligned} x - y &= 1 \\ 2x + y + 3z &= 1 \\ 2y + z &= -1. \end{aligned}$$

The solutions remain unchanged.

b. The various column operations correspond to the systems of equations

$$\text{(i)} \quad \begin{aligned} 2x + 2y + 3z &= 1 \\ x - 2y &= 1 \\ 2x + z &= 1. \end{aligned} \qquad \text{(ii)} \quad \begin{aligned} x + 2y + 3z &= 1 \\ -x + y &= 1 \\ 2y + z &= 1 \end{aligned} \qquad \text{(ii)} \quad \begin{aligned} 2x + y + z &= 1 \\ x - y + 2z &= 1 \\ 2x + z &= 1. \end{aligned}$$

For (i), the solutions are $x = 1/3, y = -1/3, z = 1/3$; i.e., the solution for y is half the original solution.

For (ii), the solutions are $x = -2/3, y = 1/3, z = 1/3$; i.e., x and y have changed places.

For (iii), the solutions are $x = 1/3, y = 0, z = 1/3$. It is rather hard to visualize what has happened. If the original system of equations was

$$x\vec{\mathbf{a}}_1 + y\vec{\mathbf{a}}_2 + z\vec{\mathbf{a}}_3 = \vec{\mathbf{b}},$$

then the new one is

$$u\vec{\mathbf{a}}_1 + v\vec{\mathbf{a}}_2 + w(\vec{\mathbf{a}}_3 - 2\vec{\mathbf{a}}_2) = \vec{\mathbf{b}}, \quad \text{i.e.} \quad u\vec{\mathbf{a}}_1 + (v - 2w)\vec{\mathbf{a}}_2 + w\vec{\mathbf{a}}_3 = \vec{\mathbf{b}}.$$

Clearly $u = x, w = z, v = y + 2w = y + 2z$ is a solution to this (in terms of the solution to the original system of equations), and that is what we found.

2.1.7 Switching rows 2 and 3 of the matrix $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ brings it to echelon form, giving $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

The matrix $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ can be brought to echelon form by multiplying row 2 by 1/2, giving $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ can be brought to echelon form by switching first the first and second rows, then the second and third rows:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $\begin{bmatrix} 0 & 1 & 0 & 3 & 0 & -3 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ can be brought to echelon form by multiplying row 2 through by -1, then adding row 3 to row 2:

$$\begin{bmatrix} 0 & 1 & 0 & 3 & 0 & -3 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

2.1.8 Suppose that A is an $n \times n$ matrix. If $\tilde{A}$ is not the identity, then there is a first diagonal term which is 0. The column containing that term has no pivotal 1, and since there are at most n pivotal 1's (at most one per row), there is some row that contains no pivotal 1. Since the first nonzero element of any row must be a pivotal one, that means that there is a row of 0's. Any row beneath a row of 0's must be a row of 0's, so the bottom row must be a row of 0's.

2.1.9 The first problem occurs when you subtract $2 \cdot 10^{10}$ from 1 to get from the second to the third matrix of equation 2.1.12 (second row, second entry). The 1 is "invisible" if computing only to 10 significant digits, and disappears in the subtraction: $1 - 20000000000 = -19999999999$, which to 10 significant digits is -20000000000. Another "invisible" 1 is found in the second row, third entry.

Solution 2.1.9: This is the main danger in numerical analysis: adding (or subtracting) numbers of very different sizes loses precision.

2.2.1 a. The augmented matrix $[A, \vec{\mathbf{b}}]$ corresponds to

$$\begin{aligned} 2x + y + 3z &= 1 \\ x - y &= 1 \\ x + y + 2z &= 1. \end{aligned}$$

Since $[A, \vec{\mathbf{b}}]$ row reduces to

$$\begin{bmatrix} \underline{1} & 0 & 1 & 0 \\ 0 & \underline{1} & 1 & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix},$$

x and y are pivotal unknowns, and z is a nonpivotal unknown.

b. If we list first the variable y, then z, then x, the system of equations becomes

$$\begin{array}{rcrcrl} y & + & 3z & + & 2x & = 1 \\ -y & & & + & x & = 1 \\ y & + & 2z & + & x & = 1. \end{array}$$

The corresponding matrix is

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}, \text{ which row reduces to } \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This time y and z are the pivotal variables, and x is the nonpivotal variable.

2.2.2 a. We have the intersection of three planes, two of which are parallel to different coordinate axes, and the third of which is parallel to none. So there is a unique solution. Indeed, row reduction gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{so } x = 1, y = 0, z = 3.$$

b. These are three planes that intersect in a point, so there is a unique solution. Indeed,

$$\begin{bmatrix} 1 & -2 & -12 & 12 \\ 2 & 2 & 2 & 4 \\ 2 & 3 & 4 & 3 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

c. Row reduction gives $\begin{bmatrix} 1 & 0 & 0 & 4.5 \\ 0 & 1 & 1 & .5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so there are infinitely many solutions, one for every choice of value for z.

d. Row reduction gives $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so there is no solution.

e. We guess (in fact we know, since the equations are certainly compatible, the zero vector is a solution) that we will get infinitely many solutions (four equations in five unknowns). Indeed, the matrix

$$\begin{bmatrix} 1 & 2 & 1 & -4 & 1 & 0 \\ 1 & 2 & -1 & 2 & -1 & 0 \\ 2 & 4 & 1 & -5 & 1 & 0 \\ 1 & 2 & 3 & -10 & 2 & 0 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can read off the solution: the variables y and w are nonpivotal, so they can be chosen freely, and the others are expressed in terms of those by

$$x = -2y + w$$
$$z = 3w$$
$$v = 0.$$

2.2.3 a. Call the equations A, B, C, D. Adding A and B gives $2x + 4y - 2w = 0$; comparing this with C gives $-2w = z - 5w + v$, so

$$3w = z + v. \tag{1}$$

Comparing C and $2D$ gives $15w = 5z + 3v$, which is compatible with equation (1) only if $v = 0$. So equation (1) gives $3w = z$.

Substituting 0 for v and $3w$ for z in each of the four equations gives $z + 2y - w = 0$.

b. Since you can choose arbitrarily the value of y and w, and they determine the values of the other variables, the family of solutions depends on two parameters.

2.2.4 For one equation in two unknowns, the simplest (and only) solution is $0x + 0y = 1$.

2.2.5 a. This system has a solution for every value of a. If you row reduce the matrix $\begin{bmatrix} a & 1 & 0 & 2 \\ 0 & a & 1 & 3 \end{bmatrix}$ you may seem to get

$$\begin{bmatrix} 1 & 0 & -1/a^2 & -\left(2/a + 3/a^2\right) \\ 0 & 1 & 1/a & 3/a \end{bmatrix},$$

which seems to indicate that there is a solution for any value of a except $a = 0$. However, obviously the system has a solution if $a = 0$; in that case, $y = 2$ and $z = 3$. The problem with the above row reduction is that if $a = 0$, it can't be used for a pivotal 1. If $a = 0$ the matrix row reduces to $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

b. We have two equations in three unknowns; there is no unique solution.

2.2.6 a. There is a solution for every value of a except $a = -6$. In the course of row reducing the matrix $\begin{bmatrix} 2 & a & 1 \\ 1 & -3 & a \end{bmatrix}$, we must multiply by $\dfrac{-2}{6+a}$, which is not possible if $a = -6$. If we continue with the row reduction, we get $\begin{bmatrix} 1 & 0 & \dfrac{6-a^2}{2(6+a)} \\ 0 & 1 & \dfrac{1-2a}{6+a} \end{bmatrix}$, which is meaningless when $a = -6$.

b. Since the first two columns of the matrix row reduce to the identity, then whenever a solution exists (whenever $a \neq -6$), the solution is unique:

$$x = \frac{6-a^2}{2(6+a)} \quad \text{and} \quad y = \frac{1-2a}{6+a}.$$

2.2.7 We can perform row operations to bring $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & a & b \\ 2 & 0 & -b & 0 \end{bmatrix}$ to

$$\begin{bmatrix} 1 & 0 & (2+a)/2 & (1+b)/2 \\ 0 & 1 & (2-a)/2 & (1-b)/2 \\ 0 & 0 & 2+a+b & 1+b \end{bmatrix}.$$

a. There are then two possibilities. If $a+b+2 \neq 0$, the first three columns row reduce to the identity, and the system of equations has the unique solution

$$x = \frac{b(b+1)}{2+a+b}, \quad y = \frac{-b^2-3b+2a}{2+a+b}, \quad x = \frac{1+b}{2+a+b}.$$

If $a+b+2=0$, then there are two possibilities to consider: either $b+1=0$ or $b+1 \neq 0$. If $b+1=0$, so that $a=b=-1$, the matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 3/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case there are infinitely many solutions: the only nonpivotal variable is z, so we can choose its value arbitrarily; the others are $x = -z/2$ and $y = 1-(3z)/2$. If $a+b+2=0$ and $b+1 \neq 0$, then there is a pivotal 1 in the last column, and there are no solutions.

b. The first case, where $a+b+2 \neq 0$, corresponds to an open subset of the (a,b)-plane. The second case, where $a=b=-1$, corresponds to a closed set. The third is neither open nor closed.

2.2.8 a. The system of equations has a solution for all values of a. In row reducing

$$\begin{bmatrix} 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \\ a & 1 & 1 & a \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

there is first a step where one must divide by $a-1$ and then a step where one must divide by $2-a-a^2$. Thus the row reduction does not apply when $a=1$ and $a=-2$. The row reduction says that there is a solution (in fact, unique solution) for every value of a except $a=1$ and $a=-2$; that solution is $x=1, y=z=0$. When $a=1$, the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \text{which row reduces to} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are no pivotal ones in the last column, so the system does have solutions; in fact, the second and third are nonpivotal, so that y and z can be chosen arbitrarily, and the $x = 1-y-z$.

Similarly in the case where $a=-2$, the augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & -2 \end{bmatrix}, \quad \text{which row reduces to} \quad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system does have roots; you can choose z arbitrarily, and then $x = 1+z,\ y=z$.

b. As discussed in part a, the system has a unique solution for every value of a except $a=1$ and $a=-2$. For those values there are infinitely many solutions: if $a=1$, each of the three equations becomes $x+y+z=1$:

one equation in three unknowns. If $a = -2$, the equations correspond to the matrix $\begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & -2 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, case 2b of theorem 2.2.1.

2.2.9 Row reducing

$$\begin{bmatrix} 1 & -1 & -1 & -3 & 1 & 1 \\ 1 & 1 & -5 & -1 & 7 & 2 \\ -1 & 2 & 2 & 2 & 1 & 0 \\ -2 & 5 & -4 & 9 & 7 & \beta \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} 1 & 0 & 0 & -4 & 3 & 2 \\ 0 & 1 & 0 & -1/3 & 7/3 & 5/6 \\ 0 & 0 & 1 & -2/3 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & \beta+1/2 \end{bmatrix}$$

There are then two possibilities: either $\beta \neq 1/2$, and there will then be a pivotal 1 in the last column (once we have divided by $\beta + 1/2$), so there is in that case no solution. If on the other hand $\beta = -1/2$, then there are infinitely many solutions: x_4 and x_5 are nonpivotal, so their values can be chosen arbitrarily, and then the values of x_1, x_2 and x_3 are given by

$$\begin{aligned} x_1 &= 2 + 4x_4 - 3x_5 \\ x_2 &= 5/6 + x_4/3 - 7x_5/3 \\ x_3 &= 1/6 + 2x_4/3 + x_5/3. \end{aligned}$$

2.2.10 Since f is invertible with differentiable inverse, we have the two compositions $f^{-1} \circ f : \mathbb{R}^n \to \mathbb{R}^n$ and $f \circ f^{-1} : \mathbb{R}^m \to \mathbb{R}^m$, whose derivatives are the identity matrix. By the chain rule, these derivatives are

$$\begin{aligned} [\mathbf{D}(f \circ f^{-1})(\mathbf{y})] &= [\mathbf{D}f(f^{-1}(\mathbf{y}))][\mathbf{D}f^{-1}(\mathbf{y})] = I \\ [\mathbf{D}(f^{-1} \circ f)(\mathbf{x})] &= [\mathbf{D}f^{-1}(f(\mathbf{x}))][\mathbf{D}f(\mathbf{x})] = I. \end{aligned}$$

Since $f(\mathbf{x}) = \mathbf{y}$ we can write $[\mathbf{D}f^{-1}(\mathbf{y})] = [\mathbf{D}f^{-1}(f(\mathbf{x}))]$; the first equation says that this $n \times m$ matrix has a left inverse and the second equation says that it has a right inverse. Therefore it is square and $n = m$.

2.2.11 a. $R(1) = 1 + 1/2 - 1/2 = 1, \qquad R(2) = 8 + 2 - 1 = 9.$

If we have one equation in one unknown, we need to perform one division. If we have two equations in two unknowns, we need two divisions to get a pivotal 1 in the first row (the 1 is free), followed by two multiplications and two additions to get a 0 in the first element of the second row (the 0 is free). One more division, multiplication and addition get us a pivotal 1 in the second row and a 0 for the second element of the first row for a total of nine.

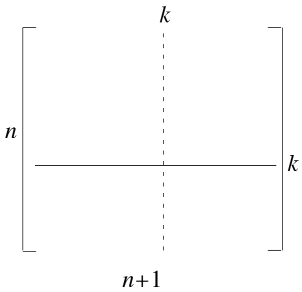

FIGURE FOR SOLUTION 2.2.11, part b: This $n \times (n+1)$ matrix represents a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of n equations in n unknowns. By the time we are ready to obtain a pivotal 1 at the intersection of the kth column (dotted) and kth row, all the entries on the kth row to the left of the kth column are 0, so we only need to place a 1 in position k, k and then justify that act by dividing all the entries on the kth row to the right of kth column by the (k, k) entry. There are $n+1-k$ such entries.

If the (k, k) entry is 0, we go down the kth column until we find a nonzero entry. In computing the total number of computations, we are assuming the worse case scenario, where all entries of the kth column are nonzero.

b. As illustrated by the figure in the margin, we need $n+1-k$ divisions to obtain a pivotal 1 in the column k. To obtain a 0 in another entry of column k requires $n+1-k$ multiplications and $n+1-k$ additions. We need to do this for $n-1$ entries of column k. So our total is

$$(n+1-k) + 2(n-1)(n+1-k) = (2n-1)(n-k+1).$$

c. For $n = 1$, we have $(2-1)(1-1+1) = 1 = 1^3 + \frac{1^2}{2} - \frac{1}{2}$, so the relationship is true for $n = 1$. If the relation is true for n, then

$$\begin{aligned}
\sum_{k=1}^{n+1}\Big(2(n+1)-1\Big)\Big((n+1)-k+1\Big) &= \sum_{k=1}^{n+1}(2n+1)(n-k+2) \\
&= 2n+1+\sum_{k=1}^{n}\Big((2n-1)(n-k+1)+(4n-2k+3)\Big) \\
&= 3n^2+4n+1+\sum_{k=1}^{n}(2n-1)(n-k+1) \\
&= 3n^2+4n+1+n^3+\frac{n^2}{2}-\frac{n}{2} = (n+1)^3+\frac{(n+1)^2}{2}-\frac{n+1}{2}.
\end{aligned}$$

So by recursion, the relation is true for all $n \geq 1$.

d.
$$\begin{aligned}
Q(1) &= \frac{2}{3}+\frac{3}{2}-\frac{7}{6} = 1, \\
Q(2) &= \frac{2}{3}8+\frac{3}{2}4-\frac{7}{6}2 = 9, \\
Q(3) &= \frac{2}{3}27+\frac{3}{2}9-\frac{7}{6}9 = 28.
\end{aligned}$$

Since $R(n) - Q(n) = \frac{1}{3}n^3 - n^2 + \frac{2}{3}n$, which is a cubic with a root at $n = 2$. Its derivative, which is $n^2 - 2n + \frac{2}{3}$, has roots at $1 \pm \sqrt{1/3}$; in particular, it is strictly positive for $n \geq 2$. So the function $R(n) - Q(n)$ is increasing as a function of n for $n \geq 2$, and hence is strictly positive for $n \geq 3$.

e. For partial row reduction for a single column, the operations needed are like those for full row reduction (part b) except that we are just putting zeros below the diagonal, so we can replace $n-1$ in the total for full row reduction by $n-k$, to get

$$(n+1-k) + 2(n-k)(n+1-k) = (n-k+1)(2n-2k+1)$$

total operations (divisions, multiplications, and additions).

f. Denote by $P(n)$ the total computations needed for partial row reduction. By part e, we have

$$P(n) = \sum_{k=1}^{n}(n-k+1)(2n-2k+1).$$

Let

$$P_1(n) = \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{6}n.$$

We will show by induction that $P = P_1$.

Clearly, $P(1) = P_1(1) = 1$. If $P(n) = P_1(n)$, we get:

$$\begin{aligned} P(n+1) &= \sum_{k=1}^{n+1}(n-k+2)(2n-2k+3) \\ &= 1 + \underbrace{\sum_{k=1}^{n}(n-k+1)(2n-2k+1)}_{P(n)} + \sum_{k=1}^{n}(4n-4k+5) \\ &= 1 + \underbrace{\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{6}n}_{P_1(n)\text{ by inductive hypothesis}} + 4n^2 - 4\frac{n^2+n}{2} + 5n \\ &= \frac{2}{3}(n+1)^3 + \frac{1}{2}(n+1)^2 - \frac{1}{6}(n+1) = P_1(n+1). \end{aligned}$$

The 1 at the beginning of the second line of this equation is the contribution from $k = n+1$.

In the third line, we get the next-to-last term using

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

So the relation is true for all $n \geq 1$.

g. We need $n-k$ multiplications and $n-k$ additions for the row k, so the total number of operations for back substitution is $B(n) = n^2 - n$.

h. So the total number of operations for n equations in n unknowns is

$$Q(n) = P(n) + B(n) = \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n \quad \text{for all } n \geq 1.$$

2.3.1 The inverse of A is $A^{-1} = \begin{bmatrix} 3 & -1 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$. Now compute

$$\begin{bmatrix} 3 & -1 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -5 \\ -1 & 1 & -2 \\ 2 & -1 & 4 \end{bmatrix}.$$

The columns of the product are the solutions to the three systems we were trying to solve.

2.3.2

a. $\begin{bmatrix} 1 & -5 \\ 9 & 9 \end{bmatrix}^{-1} = \frac{1}{54}\begin{bmatrix} 9 & 5 \\ -9 & 1 \end{bmatrix}$. b. The matrix $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$ is not invertible: subtracting 3 times the first row from the second row gives $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$.

c. $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3/2 & -1/4 & -9/4 \\ -1 & 1/2 & 3/2 \\ 1/2 & -1/4 & -1/4 \end{bmatrix}.$$

d. This matrix is not invertible: it is not square.

e. $$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 8 & 3 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 1/7 & -1/2 & 1/14 \\ 4/21 & 5/6 & -1/14 \\ -4/21 & 1/6 & 1/14 \end{bmatrix}$$ (f) $$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

g. $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

2.3.3 a. Let A be an $n \times m$ matrix. Let us first see that saying that A is invertible is the same as saying that the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for every $\vec{\mathbf{b}} \in \mathbb{R}^n$. Our definition of invertible is that A is invertible if there exists B such that $AB = I_n$ and $BA = I_m$. If you multiply through $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ from the left by B, you find

$$\vec{\mathbf{x}} = BA\vec{\mathbf{x}} = B\vec{\mathbf{b}},$$

indicating that $B\vec{\mathbf{b}}$ is the only possible solution. But is it a solution? Yes: $A(B\vec{\mathbf{b}}) = (AB)\vec{\mathbf{b}} = \vec{\mathbf{b}}$.

Now apply theorem 2.2.1 to see when the system of m equations in m variables $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for every $\vec{\mathbf{b}} \in \mathbb{R}^n$. The matrix A cannot have any nonpivotal columns, so A cannot have more columns than rows, i.e., we must have $n \le m$. But if $n < m$, then $\tilde{A}$ will definitely have a row of 0's, so there will be $\vec{\mathbf{b}}$'s for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solutions. Thus $n = m$.

b. For instance,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ but } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.3.4 a. $$A = \begin{bmatrix} 2 & 1 & 3 & a \\ 1 & -1 & 1 & b \\ 1 & 1 & 2 & c \end{bmatrix} -> \begin{bmatrix} 1 & 0 & 0 & 3a-b-4c \\ 0 & 1 & 0 & a-b-c \\ 0 & 0 & 1 & -2a+b+3c \end{bmatrix} = C$$

b. $$B^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$$

c. Multiplying A on the left by B^{-1} results in $[I_3 \quad \vec{\mathbf{h}}]$ where $\vec{\mathbf{h}}$ is the last column of C.

2.3.5 a. Since $A = \begin{bmatrix} 3 & -1 & 3 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & 3/8 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/8 \end{bmatrix}$, the solution is $x = 3/8, y = 1/2, z = 1/8$.

b. Since $\begin{bmatrix} 3 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ row reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 3/16 & 1/4 & -1/16 \\ 0 & 1 & 0 & -1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 1/16 & -1/4 & 5/16 \end{bmatrix}, \quad \text{we have}$$

$$A^{-1} = \begin{bmatrix} 3/16 & 1/4 & -1/16 \\ -1/4 & 0 & 3/4 \\ 1/16 & -1/4 & 5/16 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3/16 & 1/4 & -1/16 \\ -1/4 & 0 & 3/4 \\ 1/16 & -1/4 & 5/16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/2 \\ 1/8 \end{bmatrix}.$$

2.3.6 a. Let us row reduce:

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -5 \\ 3 & a & b \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -5 \\ 0 & a+6 & b-12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & a+b-6 \end{bmatrix}.$$

At this point, we see that the matrix is invertible if and only if $a+b \neq 6$, since in that case it row reduces to the identity.

b. Row reduce again:

$$\begin{bmatrix} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -5 & 0 & 1 & 0 \\ 3 & a & b & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -5 & 0 & 1 & 0 \\ 0 & a+6 & b-12 & -3 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 2/5 & 0 \\ 0 & 1 & -1 & 0 & 1/5 & 0 \\ 0 & 0 & a+b-6 & -3 & -(a+6)/5 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & (a+b)/(a+b-6) & 2(2a+b)/(5(a+b-6)) & -2/(a+b-6) \\ 0 & 1 & 0 & -3/(a+b-6) & (b-12)/(5(a+b-6)) & 1/(a+b-6) \\ 0 & 0 & 1 & -3/(a+b-6) & -(a+6)/(5(a+b-6)) & 1/(a+b-6) \end{bmatrix}.$$

This gives the inverse:

$$\frac{1}{a+b-6} \begin{bmatrix} a+b & 2(2a+b)/5 & -2 \\ -3 & (b-12)/5 & 1 \\ -3 & -(a+6)/5 & 1 \end{bmatrix}.$$

2.3.7 It just so happens that $A = A^{-1}$:

$$\begin{bmatrix} 1 & -6 & 3 \\ 2 & -7 & 3 \\ 4 & -12 & 5 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ So by proposition 2.3.1, the solution is

$$\vec{\mathbf{x}} = A^{-1} \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix} = A \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ -9 \end{bmatrix}.$$

2.3.8 a. The products are

$$(1)\quad \begin{bmatrix}1&0&-1\\6&3&3\\0&1&2\end{bmatrix}\quad \text{the 2nd row is multiplied by 3}$$

$$(2)\quad \begin{bmatrix}1&0&-1\\0&1&2\\2&1&1\end{bmatrix}\quad \text{the 2nd and 3rd rows are switched}$$

$$(3)\quad \begin{bmatrix}1&0&-1\\2&1&1\\2&1&0\end{bmatrix}\quad \text{twice the 1st row is added to the 3rd.}$$

b. (We use here the format for matrix multiplication introduced in section 1.2.)

$$\begin{array}{cc} & \begin{bmatrix}1&0&-1\\2&1&1\\0&1&2\end{bmatrix}\\ \begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix} & \begin{bmatrix}1&0&-1\\6&3&3\\0&1&2\end{bmatrix}\end{array},\quad \begin{array}{cc} & \begin{bmatrix}1&0&-1\\2&1&1\\0&1&2\end{bmatrix}\\ \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} & \begin{bmatrix}1&0&-1\\0&1&2\\2&1&1\end{bmatrix}\end{array},\quad \begin{array}{cc} & \begin{bmatrix}1&0&-1\\2&1&1\\0&1&2\end{bmatrix}\\ \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} & \begin{bmatrix}1&0&-1\\2&1&1\\2&1&0\end{bmatrix}\end{array}$$

c. In this case the products are

$$\begin{bmatrix}1&0&-1\\2&3&1\\3&3&2\end{bmatrix}\quad \text{the second column is multiplied by 3}$$

$$\begin{bmatrix}1&-1&0\\2&1&1\\0&2&1\end{bmatrix}\quad \text{the second and third columns are switched}$$

$$\begin{bmatrix}-1&0&-1\\4&1&1\\4&1&2\end{bmatrix}\quad \text{twice the third column is added to the first.}$$

Multiplying out gives the same result:

$$\begin{array}{cc} & \begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}\\ \begin{bmatrix}1&0&-1\\2&1&1\\0&1&2\end{bmatrix} & \begin{bmatrix}1&0&-1\\2&3&1\\3&3&2\end{bmatrix}\end{array},\quad \begin{array}{cc} & \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\\ \begin{bmatrix}1&0&-1\\2&3&1\\0&3&2\end{bmatrix} & \begin{bmatrix}1&-1&0\\2&1&1\\0&2&1\end{bmatrix}\end{array},\quad \begin{array}{cc} & \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\\ \begin{bmatrix}1&0&-1\\2&1&1\\0&1&2\end{bmatrix} & \begin{bmatrix}-1&0&-1\\4&1&1\\4&1&2\end{bmatrix}\end{array}.$$

2.3.9 a. The products will be

$$\begin{bmatrix} -2 & 3 & -14 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{3 times the third row is subtracted from the first}$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & 6 \\ 1 & 0 & 4 \end{bmatrix} \quad \text{the second row is multiplied by 2}$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix} \quad \text{the second and third rows are switched.}$$

b.

$$\begin{array}{cc} & \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} -2 & 3 & -14 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & 6 \\ 1 & 0 & 4 \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 3 & -2 \\ 1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix} \end{array}.$$

2.3.10 a. Clearly, $E_2(i,j,x)A$ will have the same rows as A except for the ith. The ith row of $E_2(i,j,x)A$ is the sum of the ith row of A, contributed by the 1 in position (i,i), and of x times the jth row of A, contributed by the x in position (i,j).

b. The rows of $E_3(i,j)A$ are those of A, except for the ith. The ith row of $E_3(i,j)A$ is the jth row of A, contributed by the 1 in the (i,j)th position, and similarly the jth row of $E_3(i,j)A$ is the ith row of A

2.3.11 Let A be an $n \times m$ matrix. Then

$AE_1(i,x)$ has the same columns as A, except the ith, which is multiplied by x.

$AE_2(i,j,x)$ has the same columns as A except the jth, which is the sum of the jth column of A (contributed by the 1 in the (j,j)th position), and x times the ith column (contributed by the x in the (i,j)th position).

$AE_3(i,j)$ has the same columns as A, except for the ith and jth, which are switched.

2.3.12 The k,lth entry of $E_1(i,x)E_1(i,1/x)$ is

$$\begin{cases} \vec{\mathbf{e}}_k \cdot \vec{\mathbf{e}}_l = 0 & \text{if } k,l \neq i \text{ and } k \neq l \\ \vec{\mathbf{e}}_k \cdot \vec{\mathbf{e}}_k = 1 & \text{if } k,l \neq i \text{ and } k = l \\ x\vec{\mathbf{e}}_i \cdot \vec{\mathbf{e}}_l = 0 & \text{if } k = i, l \neq i \\ \vec{\mathbf{e}}_k \cdot (1/x)\vec{\mathbf{e}}_i = 0 & \text{if } k \neq i, l = i \\ x\vec{\mathbf{e}}_i \cdot (1/x)\vec{\mathbf{e}}_i = 1 & \text{if } k = l = i. \end{cases}$$

These are the entries of the identity matrix.

Now for $E_2(i,j,x)$. Let us set this up in our standard way:

$$\begin{array}{cc} & \begin{array}{cc} i & j \\ \end{array} \\ & \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \\ \begin{array}{c} i \\ j \end{array} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -x+x \\ 0 & 1 \end{bmatrix} \end{array}$$

Finally, let us check for $E_3(i,j)$. Again, just set up the multiplication:

$$\begin{array}{cc} & \begin{array}{cc} i & j \\ \end{array} \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{array}{c} i \\ j \end{array} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

2.3.13 Here is one way to show this. Denote by a the ith row and by b the jth row of our matrix. Assume we wish to switch the ith and the jth rows. Then multiplication on the left by $E_2(i,j,1)$ turns the ith row into $a+b$. Multiplication on the left by $E_2(j,i,-1)$ then by $E_1(j,-1)$ turns the jth row into a. Finally, we multiply on the left by $E_2(i,j,-1)$ to subtract a from the ith row, making that row b. So we can switch rows by multiplying with only the first two types of elementary matrices.

Here is a different explanation of the same argument: Compute the product

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This certainly shows that the 2×2 elementary matrix $E_3(1,2)$ can be written as a product of elementary matrices of type 1 and 2.

More generally,

$$E_2(i,j,-1)E_1(j,-1)E_2(j,i,-1)E_2(i,j,1) = E_3(i,j).$$

2.3.14 a. First multiply on the left by a type 2 elementary matrix to add -4 times the 1st (jth) row to the 2nd (ith row). Second, multiply by a type 1 elementary matrix to multiply the second (ith) row by $-1/3$; third, multiply again by a type 2 elementary matrix to add -2 times the 2nd (jth) row to the 1st (ith) row:

$$\begin{array}{cc}
 & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}
\end{array}$$

(We use here the format for repeated multiplication already used in Section 1.2.)

b. To save on tedium, first we multiply by a modified type 2 matrix to add 1 times the first row to both the second and third rows. Next, we multiply by a type 1 elementary matrix to multiply the second (ith) row by -1. Third, we add 1 times the 2nd (jth) row to the 1st (ith) row, using a type 2 elementary matrix. Fourth, we multiply the third row by $1/2$ using a type 1 elementary matrix. Last, we use a modified type 2 matrix to add 2 times the 3rd row to the 1st row, and 3 times the 3rd row to the 2nd row:

$$\begin{array}{cc}
 & \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} & \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}.$$

c.

$$\begin{array}{cc}
 & \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 0 & -1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & -6 & -11 \\ 0 & 1 & 2 & 3 \end{bmatrix} \\
\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -9 & -17 \\ 0 & -1 & -6 & -11 \\ 0 & 0 & -4 & -8 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} & \begin{bmatrix} 1 & 0 & -9 & -17 \\ 0 & 1 & 6 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\end{array}.$$