Math 8201 **Final Homework** **PJW**

Date due: I announced **4pm Tuesday December 19, 2017** and that is not a bad deadline but, to be honest, if you need more time I would accept that.

There are five questions altogether, each worth 20%. Give careful and complete arguments. You may quote without proof any results that appear in the book by Dummit and Foote, any results from homework questions that you have been assigned, and any results that were lectured in class, provided that you indicate that you are doing this and that **they do not invalidate the question** (in my opinion). No other results may be quoted. Relevant arguments found in any book may be used, but should not be copied word for word. The work should be your own – please do not consult other people. If you want me to clarify the meaning of questions I will do that, but I will not give anyone hints. I can be contacted by email: webb@math.umn.edu; or my office telephone: (612) 625 3491; or my home telephone: (507) 645 8150; and if you happen to find me in my office, that works too! The following questions all appeared on the graduate written exam, rather a long time ago. This homework will count for 16% of your total score, if you do it.

A. (Fall 2001, question 1) (20%)
Let $G = GL(2,3)$ be the group of 2×2 invertible matrices with entries in the field $\mathbb{F}_3$ with 3 elements. Let P be any Sylow 3-subgroup of G and let $N_G(P)$ be its normalizer.
(a) (14) Prove that $N_G(P)$ is conjugate to the subgroup

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{F}_3,\ a \neq 0 \neq c \right\}.$$

(b) (6) Show that $N_G(P)$ is not isomorphic to the alternating group A_4.

B. (Fall 1994, question 6) (20%)
Prove that a finite group is cyclic if and only if for every $k \geq 1$ it contains at most k elements of order dividing k.
[First prove the result for groups of prime power order and then use Sylow's theorems.]

C. (Spring 2000, question 1) (20%) Let G be a group whose order is a power of a prime number, say p^n. Prove for each i between 0 and n, that G has a subgroup of order p^i.

PLEASE TURN OVER.

D. (Spring 1995, question 7) (20%) For each prime number p put

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z},\ p \not| b\}$$

as a subring of $\mathbb{Q}$. Show that $\mathbb{Z}_{(p)}$ has a unique non-zero prime ideal. Show further that if $p \neq q$ are distinct primes then $\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ has just two non-zero prime ideals, each of which is maximal.

E. (Spring 2001, question 6) (20%) Let K be a field.

(a) (7) Show that for every element $a \in K$, the rings $K[X]/(X^2)$ and $K[X]/((X-a)^2)$ are isomorphic.

(b) (13) Let A be any ring with a 1 that contains K as a subring (containing the 1), and suppose that as a vector space over K, $\dim(A) = 2$. Show that either $A \cong K[X]/(X^2)$ or $A \cong K \times K$ or A is a field.