Math 8202 **Homework 3** **PJW**
Date due: February 12, 2018

Hand in only the starred questions.

At this point we will only do the first three sections from Chapter 10. Section 10.4 is about tensor products, which are not on the syllabus. Section 10.5 is about projective and injective modules, which are on the syllabus, but I propose to move straight on Section 12.1 after 10.3 and come back to 10.5 later. I will assume that you know the linear algebra in Chapter 11 up to section 11.4 (inclusive).

We are also going to do the Jordan-Hölder theorem for modules at this point, which is not covered in Dummit and Foote, but is on the syllabus.

Section 10.3 nos. 5, 7*, 8, 9, 10, 11*, 12, 14*, 15.

A*. (Fall 2000, qn. 3) Let S be the ring $\mathbb{R}[X]/(X^2+1)^3$. Prove that the free S-module $S \oplus S$ has a composition series. Find the length of a composition series of $S \oplus S$.

B*. (Spring 2001, qn. 5) Let M be a module over a ring R which has a composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M.$$

(a) Suppose that the modules in this composition series form a complete list of all the submodules of M. Show that it is not possible to write M as a direct sum of submodules $M = N_1 \oplus N_2$ unless either $N_1 = 0$ or $N_2 = 0$. Show further that every submodule of M can be generated by a single element.

(b) Show by example that it is possible to have a module M which simultaneously has the properties

- every submodule of M can be generated by a single element,
- M has a composition series, and
- there are submodules of M other than the ones in the composition series.

From a while back, the following question on composition series of modules was also set. We cannot do it yet because we have not studied the Noetherian property.

C. (Spring 2002, qn. 4) (12%) Let R be a commutative Noetherian ring with a 1 and I a maximal ideal of R.

(a) (6%) Show that if M is a finitely generated R-module then M/IM has finite composition length as an R-module.

(b) (6%) Show that R/I^{10} has finite composition length as an R-module.