Math 8246 **Homework 1** **PJW**

Date due: Wednesday February 27, 2019.

1. (Like D&F 17.1, 8) Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ the sequence $0 \to \mathrm{Ext}^n_R(D,L) \to \mathrm{Ext}^n_R(D,M) \to \mathrm{Ext}^n_R(D,N) \to 0$ is also short exact and split. [Use a splitting homomorphism and the fact that Ext is functorial in each variable.]

2. (a) Let M and N be $\mathbb{Z}G$-modules and suppose that N has the trivial G-action. Show that $\mathrm{Hom}_{\mathbb{Z}G}(M,N) \cong \mathrm{Hom}_{\mathbb{Z}G}(M/(IG\cdot M),N)$.
(b) Show that for all groups G, $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},IG) = 0$; and that if we suppose that G is finite then $\mathrm{Hom}_{\mathbb{Z}G}(IG,\mathbb{Z}) = 0$.
(c) By applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(IG,\quad)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ show that for all finite groups G, if $f : IG \to \mathbb{Z}G$ is any $\mathbb{Z}G$-module homomorphism then $f(IG) \subseteq IG$.
(d) Show that if G is finite and $d : G \to \mathbb{Z}G$ is any derivation then $d(G) \subseteq IG$. Is the same true for arbitrary groups G?

3. Let G be a finite group. Show that the endomorphism ring $\mathrm{Hom}_{\mathbb{Z}G}(IG,IG)$ is isomorphic to $\mathbb{Z}G/(N)$ where $N = \sum_{g\in G} g$ is the norm element which generates $(N) = (\mathbb{Z}G)^G$.
[You may assume that every $\mathbb{Z}G$-module homomorphism $IG \to \mathbb{Z}G$ has image contained in IG. Apply the functor $\mathrm{Hom}_{\mathbb{Z}G}(-,\mathbb{Z}G)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$. You may assume for a finite group G that $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}G) = 0$.]

4. Show that for every group G:
(a) all derivations $d : G \to M$ satisfy $d(1) = 0$, and
(b) the mapping $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$ is a derivation.

5. (a) Show that the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ is split as a sequence of $\mathbb{Z}G$-modules if and only if $G = 1$. Deduce that the identity group is the only group of cohomological dimension 0.
(b) Show that if G is a free group then $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}G) \neq 0$.

6. If N is a right $\mathbb{Z}G$-module and M is a left $\mathbb{Z}G$-module we may make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n\otimes m) = ng^{-1} \otimes gm$, extended linearly to the whole of $N \otimes_{\mathbb{Z}} M$. Show that $N \otimes_{\mathbb{Z}G} M \cong (N \otimes_{\mathbb{Z}} M)_G$.
[Not part of the question, just information: if N and M are two left modules we make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = gn \otimes gm$. This is called the *diagonal action* on the tensor product.]